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ELEMENTARY MODERN GEOMETRY PART I.

(MATRICULATION GEOMETRY)

BOOKS I—IV. (Complete.)

Meeting the requirements of the Matriculation Examination prescribed under the new regulations of the Calcutta University.



BY

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AND

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"Intermediate Solid Geometry" &c. &c.*

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PREFACE.

“Elementary Modern Geometry, Part I.” (Matriculation Geometry) comprises Books I. to IV., which form the Matriculation Course, Part II. containing Book V. (Solid Geometry) only. Each Book has been treated on the lines of the syllabus prescribed under the new regulations of the Calcutta University, and I have spared no pains to keep a steady eye on simplicity and clearness at every stage of the student’s progress. Books I., II., III., with the exception of Sections III, and IV. of Book III., may be generally taken to include the Compulsory Course for the Matriculation Examination, whilst Books I. to IV. may be taken to include the Optional (or Additional) Course.

The order in which Propositions are numbered is not the same in every treatise on Modern Geometry. Hence, in all answers meant for the Examiner, the student, while giving a reference, should *never* quote the number of the Proposition referred to, but simply point out its application with clearness and precision. For instance, in proving Theorem 14, Book III, Theorem 10 of the same Book has to be referred to, but all that the student need do is to write out the steps, as given in this treatise, taking care to omit “(Th. 10)” at the end of “ \therefore the $\angle ADC =$ half the $\angle AOC$.”

Model Papers have been inserted at the end of the treatise. The student is strongly recommended to attempt to answer each Paper within the time allotted, and measure his progress by the amount of success attained in such endeavours. It is also advisable that each of these Papers should be answered oftener than once, for a gradually increasing speed and facility may be acquired only by frequent trials.

Any suggestions for the improvement of the work will be thankfully received.

DACCA : January, 1911.

K. P. BASU.

PREFACE TO THE FIFTH EDITION.

In this Edition, on account of better arrangement and printing the book has increased in volume but in accordance with the wishes of the Hon'ble the Director of Public Instruction, Bengal, the price has been lowered from Re. 1-8 to Re. 1-4 for Books I.—IV. (*Complete*). Books I. & II. and III. & IV. being separately priced at As. 10 and As. 12 respectively. There has also been some improvement in the contents by inserting short references to the various propositions. It is, therefore, hoped that this Edition will be found more useful and acceptable than the previous ones. It is also to be noted that this book has been approved by the Government of Bengal as a Text, Prize and Library Book.

11, *Mohendra Gossain Lane, Calcutta,* }
January, 1916. }

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ELEMENTARY

MODERN GEOMETRY

BOOK I.

ANGLES AT A POINT ; PARALLEL STRAIGHT
LINES ; TRIANGLES AND RECTILINEAL FIGURES.

SECTION I.

FUNDAMENTAL IDEAS AND DEFINITIONS.

1. That which has length, breadth and thickness is called a **solid**.

Note 1. Each of the following pictures represents a *solid* :—



Note 2. The first figure represents a brick-shaped body, whilst the second represents a ball. The length, breadth and thickness of the former are easily understood, whilst those of the latter are not. But a ball may be easily conceived to be mostly made up of parts, large and small, which are all brick-shaped. Thus it is clear that anything which may be conceived to be mostly composed of brick-shaped parts is a solid.

Note 3. From the above it is also clear that the hollow portion of a box, quite as much as the entire box, is a solid. In fact, any

definite portion of *space* possessing, as it does, length, breadth and thickness, is a **solid**.

2. That which has length and breadth, but *no thickness*, is called a **surface**.

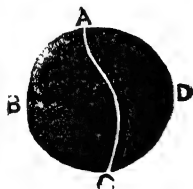
Note 1 A solid is bounded by a *surface* or *surfaces*. For instance, consider a brick-shaped piece of wood. It may be said to have six boundaries, every one of which separates the wood from the rest of space. Each boundary evidently possesses length and breadth, but it has *no thickness*; for, if it had any, it would undoubtedly include a slice of the wood or a thin layer of the surrounding air, or both, which it *does not*. Similarly, the boundary of a ball-shaped body has *no thickness*, because the boundary neither forms part of the body nor that of the surrounding air; but it *has* length and breadth, in the sense that it may be supposed to be divided into small portions, each of which is similar in shape to one or other of the boundaries of a brick-shaped body. The boundary between two portions of the same solid is also a *surface*.

Note 2. It is evident therefore that a surface occupies *no space*.

Note 3. If an ink-spot were made on paper like this would it form a solid or a surface? Undoubtedly it would form a *solid*, because it has not only length and breadth but also *thickness*, however small the thickness may be. Two surfaces of the ink-spot are very close to each other, one (the top-surface) separates the spot from the surrounding air, and the other (the bottom-surface) separates it from the paper.

3. That which has length only, but *neither breadth nor thickness*, is called a **line**

Note 1. The boundary between two portions of a surface is a *line*. Let **ABCD**, in the accompanying diagram, represent a surface and let the white mark **AC** represent the boundary between the portions **ABC** and **ADC**.



I.] FUNDAMENTAL IDEAS AND DEFINITIONS. 3

So long as the white mark has any breadth at all, the two portions **ABC** and **ADC** form two *distinct* surfaces, one on either side of it ; it is only when its breadth dwindles down to *nothing* that the surfaces **ABC** and **ADC** become *parts* of the given surface (*i.e.* they together make up the whole surface). Thus the boundary between two parts of a surface has length only and *no breadth*, and hence it is a *line*.

Note 2. It is also clear that any given surface is bounded by a line or lines.

Note 3. If an ink-mark like this———be made on paper, it is *not* a line according to our definition, for it has some breadth and also a little thickness. The finer the mark the more nearly does it *resemble* a line.

4. That which has position only, but *neither length nor breadth, nor thickness*, is called a **point**.

Note 1. The boundary between two parts of a given line is a point ; because, if this boundary had any length, the two parts would be two distinct lines instead of being parts of the given line. It is clear also that the extremities of a line are points.

Note 2. The picture of a point is a dot (.) ; but it must be remembered that a dot is in reality a *solid* and *not* a point. The smaller the dot the more nearly does it *resemble* a point.

Note 3. A portion of a line, however small, can *not* be regarded as a point ; for, such a portion has some length, whilst a point has none. All that we can say is that the smaller the portion the more nearly does it approach the condition of a point.

Note 4. If a point moves from one position to another very close to it, a small line is generated, of which the two positions of the point are the extremities.

5. Of the three things—length, breadth and thickness, each is called a **dimension**. That which possesses one or more of the three dimensions is called a **magnitude**.

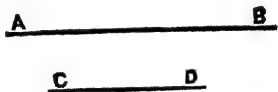
Note 1. Hence a solid is said to be a magnitude of *three* dimensions ; a surface is said to be a magnitude of *two* dimensions ; a line is said to be a magnitude of *one* dimension ; a point is not a magnitude, and it has *no* dimension.

Note 2. The word magnitude is used also in the sense of *size* or *extent*, as in the expression "A point is that which has no parts, or which has no magnitude."

6. If two lines be such that whenever they coincide in two points they invariably become indistinguishable from one another, each of them is called a **straight line** or a **right line**.

Let AB, CD be two lines, as in the following diagram.

Suppose the line CD to be taken up and placed in such a way that the points C and D fall upon the line AB ; on whatever points of the line AB the points C and D may fall, if in every case the two lines become indistinguishable



one from the other, then *each* of the lines AB and CD is a straight line. In the following diagram the two lines AB and CD are quite distinguishable from one another even though they coincide in two points :—

In this case therefore the lines AB and CD are *not* straight lines.



Note 1. From the above definition the following conclusions are evident :—

(1) Two straight lines cannot have more than one point common ; for, if they had two points common, they would completely become one and the same straight line. This is otherwise expressed

1.] FUNDAMENTAL IDEAS AND DEFINITIONS. 5

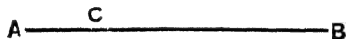
by saying that *two straight lines cannot intersect each other in more points than one*.

(2) Two straight lines cannot coincide partly without being altogether indistinguishable from one another; for, if they coincide partly, they certainly coincide in two points, and therefore they must be indistinguishable throughout. This fact is expressed by saying that *two straight lines cannot have a common segment* (a *segment* of a line meaning a *part* of it).

(3) Through one given point an unlimited number of straight lines may pass but *only one straight line can pass through two given points*.

Note 2. A straight line whose extremities are given is called a **finite** or **limited** straight line; whilst one which may be supposed to be extended to any length either way is called an **unlimited** straight line.

Note 3. Let **AB** be a finite straight line whose extremities are the points **A** and **B**.



Let a point **C** start from **A** and move along the line towards **B**. In any position, **C** will divide the line into two parts **AC** and **CB**, and it is quite clear that as **C** moves on, the part **AC** gradually increases whilst **CB** gradually diminishes. Hence there must be one position of **C**, and *one only*, for which the parts **AC** and **CB** will be equal. In this position **C** is said to **bisect** the line **AB**, and it is called the middle point, or, more shortly, the **mid-point** of **AB**. Hence the mid-point of a given straight line may be defined to be the point which divides it into two equal parts.

Note 4. Let **AB** and **CD** be two finite straight lines.



Suppose **AB** is taken up and placed in such a way that the point **A** falls on the point **C** and the line **AB** upon **CD**; if now **B**

also falls on **D**, then the line **AB** *coincides* with **CD** and is therefore equal to it. Hence we may say that *one finite straight line is equal to another when either of them may be made to coincide with the other.*

Note 5. A line which is neither straight nor made up of straight lines is called a **curved line**, or, more simply, a **curve**. The accompanying diagram is the picture of a curved line.



Note 6. When a point moves along a straight line its direction of motion remains unchanged, whilst if a point is moving along a curve its direction of motion constantly changes. In the accompanying pictures the arrow heads shew the directions of motion.



7. If a surface be such that the straight line passing through *any* two points in it lies wholly in that surface, it is called a **plane surface**, or, more simply, a **plane**.

Note 1. If the surface be unlimited all round, the straight line passing through the two points should also be considered as unlimited in both directions. This unlimited straight line then, for all position of the points, will be *completely* in contact with the surface, *if it is a plane*.

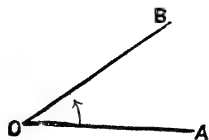
Note 2. If two points be taken on the surface of a ball-shaped body, the straight line passing through them does not at all lie in that surface. Clearly therefore such a surface *is not* a plane surface.

Note 3. Henceforth, unless the contrary is stated, *all points and lines shall be supposed to be existing in one and the same plane.*

8. If two straight lines **OA** and **OB** meet at a point **O**, the straight line **OB** may be supposed to have originally

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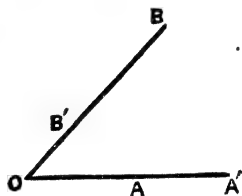
coincided in direction with OA , and then to have moved from that position into its present one by a motion of rotation about the point O ; it is this *amount of rotation* which is called the **angle** contained by the straight lines OA and OB .



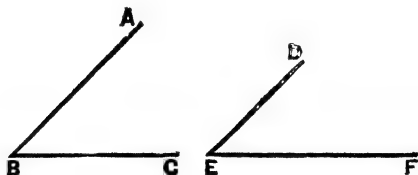
Note 1. The angle contained by the straight lines OA and OB is briefly expressed as “the angle AOB ”, or “the angle BOA ”, the letter O being always put in the middle.

Note 2. The point O is called the **vertex** of the angle; and the straight lines OA and OB are called its **arms**.

Note 3. The size of an angle does not at all depend upon the *lengths* of its arms. Hence, in the accompanying diagram, the angle AOB is the same as the angle $A'OB'$.

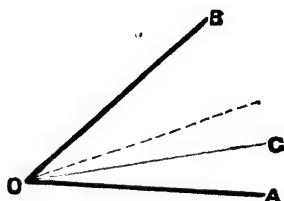


Note 4. Let there be two angles ABC and DEF .



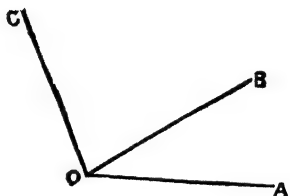
Suppose the angle ABC to be taken up, with its size unchanged, and so placed that the point B falls on the point E , and BC upon EF ; BA and ED being on the same side of EF . If now BA also falls upon ED , then the angle ABC *coincides* with the angle DEF and is therefore *equal* to it. Hence *two angles are equal if either of them may be made to coincide with the other*. It is also clear that BA will fall on the left or on the right of ED according as the angle ABC is greater or less than the angle DEF .

Note 5. Let AOB be an angle ; and let a straight line OC starting from a position of coincidence with OA rotate about O towards OB .

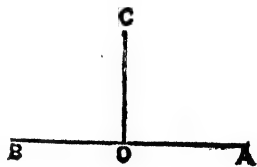


As OC goes on rotating, the angle AOC gradually increases whilst the angle COB gradually decreases. Clearly therefore there must be one position of OC , and *one only*, in which the angle AOC is *equal* to the angle COB . In this position OC is said to **bisect** the angle AOB , and is called the **bisector** of the angle. Hence the angle AOB , and similarly every other angle, *has one and only one bisector*.

Note 6. If three straight lines OA , OB , OC meet at a point O , OB occupying an intermediate position, then the angles AOB and BOC are called **adjacent angles**. In other words, two angles are said to be *adjacent* when they have a common vertex and lie on opposite sides of a common arm. The two extreme lines OA and OC may be conveniently called the **non-coincident arms** of the two angles.

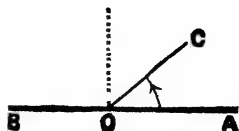


9. If a straight line OC standing upon another straight line AOB make the adjacent angles AOC , COB equal to one another, each of these angles is called a **right angle** ; and each of the two straight lines is said to be **perpendicular** to the other.



I.] FUNDAMENTAL IDEAS AND DEFINITIONS. 9

Note 1. Let AOB be any straight line, and OC another which starting from the position OA rotates about O in the direction of the arrow-head, as shewn in the accompanying diagram.

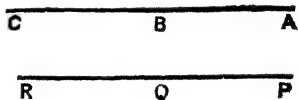


As OC goes on rotating, it is evident that the angle AOC gradually increases whilst the angle COB gradually decreases. Clearly therefore there is one position of OC , and *one only*, for which the angle AOC is equal to the angle COB ; and it is in this position that OC is perpendicular to AB . Hence at any given point in a straight line there can be *only one perpendicular* to it.

Note 2. As OC rotates about the point O from the position OA into the position OB , it turns through two right angles. Hence, when the straight lines OA and OB are in the same straight line but in *opposite directions*, the angle AOB is equal to *two right angles*.

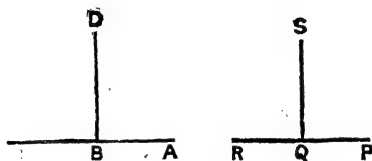
Note 3. When OA and OB are in the same straight line but in opposite directions, the angle AOB is called a **straight angle**. In other words, an angle is called a straight angle *when its arms are in the same straight line but in opposite directions*, and a *straight angle is equal to two right angles*. Hence a *right angle is half of a straight angle*.

Note 4. Let ABC and PQR be any two straight lines.



Suppose the straight line ABC to be taken up and so placed that the point B falls on the point Q , and BA upon QP . Then BC must also fall upon QR ; because otherwise the two straight lines would have a common segment. Thus it is clear that if ABC and PQR be two *straight angles*, they can be made to coincide. Hence the straight angle PQR is equal to the straight angle ABC . Similarly every other straight angle is equal to the straight angle ABC . Clearly therefore *all straight angles are equal to one another*. Hence as a right angle is half of a straight angle, *all right angles are equal*.

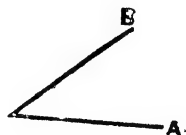
Note 5. That all right angles are equal may also be proved in the following manner. Let the straight line **BD** be perpendicular to the straight line **ABC**, and let the straight line **QS** be perpendicular to the straight line **PQR**. Then we have two pair of right angles **ABD**, **CBD** and **PQS**, **RQS**.



Suppose the first figure to be taken up, without any alteration in its shape, and so placed upon the second that the point **B** falls upon the point **Q**, and the line **ABC** upon the line **PQR**; and so that **BD** and **QS** are on the same side of **PQR**. In this circumstance **BD** and **QS** both become perpendicular to **PQR** at **Q**; therefore **BD** must fall upon **QS**, as there can be but *one* perpendicular to a straight line at any point in it. Hence the angle **ABD** is equal to the angle **PQS**, and the angle **CBD** to the angle **RQS**. Hence it is clear that each angle of the first pair is equal to each angle of the second. Similarly, to whichever pair of right angles a particular right angle may belong, this latter must be equal to one or other of the angles **PQS** and **RQS**. Hence, *all right angles are equal to one another*.

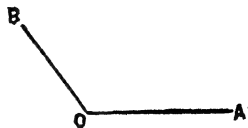
Note 6. When we say that **OC** is *at right angles* to **OA** we mean that **OC** is *perpendicular* to **OA**.

10. An angle which is less than a right angle is called an **acute angle**.

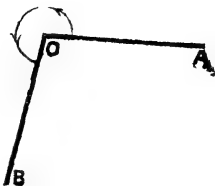


I.] FUNDAMENTAL IDEAS AND DEFINITIONS. 11

11. An angle which is greater than one right angle but less than two right angles, is called an **obtuse angle**.



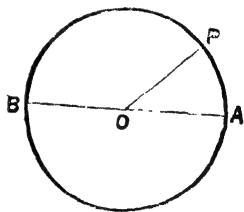
Note. An angle which is greater than two right angles but less than four right angles, is called a **reflex** or **re-entrant angle**.



12. Any portion of a plane surface bounded by one or more lines is called a **plane figure**.

Note. In a more extended sense, any combination of points and lines is called a *figure*. The picture of a figure is called a *diagram*. Real points and lines can never be constructed; so whatever is drawn on paper is only a *picture* of the real thing. From this it is clear that when a figure is drawn on paper it is the diagram that is really drawn, and the finer the diagram the more nearly does it represent the figure intended.

13. A **circle** is a plane figure bounded by one line which is such that all straight lines drawn to it from a certain point within the figure are equal to one another.



14. The bounding line of a circle is called its **circumference**.

15. The point within a circle from which all straight lines drawn to the circumference are equal, is called the **centre** of the circle.

16. Any straight line drawn from the centre of a circle to the circumference is called a **radius** of the circle.

Note. All radii of the same circle are equal to one another.

17. Any straight line drawn through the centre of a circle and terminated both ways by the circumference is called a **diameter** of the circle.

18. A **semi-circle** is the figure bounded by a diameter of a circle and the part of the circumference which it cuts off.



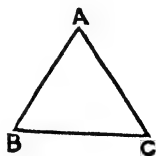
Note. Every diameter divides a circle into two semi-circles.

19. A plane figure which is bounded by straight lines only is called a **plane rectilineal figure**.

Note 1. The bounding straight lines are called the **sides** of the figures.

Note 2. The amount of surface enclosed by the boundaries of a plane figure is called its **area**.

20. A plane figure bounded by *three* straight lines is called a **triangle**.



Note 1. Generally speaking, any angular point of a triangle is called a **vertex** of the triangle, so that a triangle has *three vertices*. But when a particular side is spoken of as the **base** of the triangle,

I.] FUNDAMENTAL IDEAS AND DEFINITIONS. 13

it is then the opposite angular point alone that is called the vertex. Thus, in the above diagram, when the side **BC** is regarded as the base of the triangle, the angular point **A** is called the vertex.

Note 2. In any triangle the straight line drawn from a vertex to the middle point of the opposite side is called a **median**.

21. A triangle whose three sides are equal is called an **equilateral triangle**.



22. A triangle which has two sides equal is called an **isosceles triangle**.

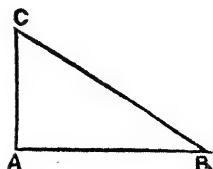


Note. In an isosceles triangle the point of intersection of the equal sides is generally taken as the vertex of the triangle, and the opposite side as the base. The angle contained by the equal sides is called the *vertical angle* of the triangle.

23. A triangle which has three unequal sides is called a **scalene triangle**.



24. A triangle one of whose angles is a right angle is called a **right-angled triangle**.



Note. In a right-angled triangle, the side opposite to the right angle is called the **hypotenuse**. Thus, in the above diagram, the side **BC** is the hypotenuse of the triangle **ABC** which is right-angled at **A**.

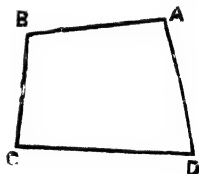
25. A triangle one of whose angles is an obtuse angle is called an **obtuse-angled triangle**.



26. A triangle of which all the angles are acute is called an **acute-angled triangle**.



27. A plane figure bounded by *four* straight lines is called a **quadrilateral**.



Note. A straight line joining two opposite angular points of a quadrilateral is called a **diagonal** of the quadrilateral. Thus, in the above diagram the straight line joining **A** and **C** is a diagonal, and so is the straight line joining **B** and **D**.

28. A plane figure bounded by *more than four* straight lines is called a **polygon**.

29. Straight lines which are in the same plane and which do not meet, however far they may be produced either way, are called **parallel straight lines**.



30. A quadrilateral whose opposite sides are parallel is called a **parallelogram**.

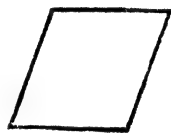


31. A parallelogram of which one angle is a right angle is called a **rectangle**.



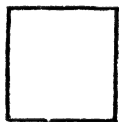
Note. It will be proved hereafter that if one angle of a parallelogram be a right angle *all* its angles are right angles.

32. A parallelogram of which two adjacent sides are equal, but whose angles are not right angles, is called a **rhombus**.



Note. It will be proved hereafter that if two adjacent sides of a parallelogram be equal, *all* its sides are equal.

33. A rectangle of which two adjacent sides are equal, is called a **square**.



34. A quadrilateral which has one pair of opposite sides parallel, is called a **trapezium**.



SECTION II.

POSTULATES AND HYPOTHETICAL CONSTRUCTIONS.

1. For the purpose of effecting geometrical constructions, certain constructions that may be easily admitted as possible are made the foundation of others which are not so obvious. These simple and easily admissible constructions are called **postulates**. They are three in number :—

(1) *A straight line may be drawn from any one point to any other point.*

(2) *A finite (or terminated) straight line may be produced to any length in that straight line.*

(3) *A circle may be drawn with any point as centre and with a radius equal to any finite straight line.*

Note. The first two constructions may be practically carried out with the help of an ungraduated flat ruler, whilst the third requires the use of a pair of compasses which may be so adjusted as to transfer a distance from one position to another.

2. Some constructions which are not so obvious as the postulates, and which, in fact, may have to be performed with the help of one or more of them, may as well be *taken for granted* for the purpose of establishing a geometrical truth. A construction, which is thus assumed before it is shewn *how* it can be performed, is called a **hypothetical construction**. It is clear therefore that a construction which is hypothetical.

in one place is not necessarily so in another. In Book I. the following constructions will be assumed before they are formally proved :—

(1) A perpendicular may be drawn to a given straight line from any given point in that straight line.

(2) A straight line may be drawn to bisect any given angle.

(3) Through any given point a straight line may be drawn parallel to a given straight line.

SECTION III.

AXIOMS (OR SELF-EVIDENT TRUTHS).

1. Things which are equal to the same thing are equal to one another.
2. If equals be added to equals the wholes are equal.
3. If equals be taken from equals the remainders are equal.
4. If equals be added to unequals the wholes are unequal.
5. If equals be taken from unequals the remainders are unequal.
6. Things which are doubles of the same thing, or of equal things, are equal to one another.
7. Things which are halves of the same thing, or of equal things, are equal to one another.
8. Magnitudes which can be made to coincide with one another are equal.

Note 1. It is assumed that any magnitude may be taken up from its position and placed upon another magnitude *without change of shape or side*.

Note 2. The word magnitude is to be understood to mean a line or an angle, or a figure.

Note 3. The process of taking up one magnitude from its position and placing it upon another is called **superposition**; and the former magnitude is said to be **applied** to the latter.

9. The whole is greater than its part.

N.B. It is also obvious that the whole is equal to the sum of its parts.

10. Two straight lines which coincide in their extremities coincide completely.

N.B. This is equivalent to saying that "two straight lines cannot enclose a surface".

11. All right angles are equal.

12. Two straight lines which intersect one another, cannot be both parallel to the same straight line.

(Playfair's Axiom.)

EXERCISE (1).

1. Is a very small ink-spot on paper a point? if not, what is it?

2. Is a straight line drawn on paper really a geometrical *straight line*? If not, what is it?

3. If a finite straight line be supposed to be divided into a very large number of equal parts, has any of these parts any *dimension*? If so, what?

4. What kind of magnitude is the *boundary* between the calm water of a tank and the air above? and why?

5. Define a *plane*. If on a finite plane two parallel straight lines be supposed to be drawn *very close* to each other, and to be terminated by the boundary or boundaries of the plane, does any portion of the plane lie between the two lines? If so, under what circumstances will there be *no* portion of the surface between them?

6. If two plane surfaces have two points A and B in common, prove that the straight line drawn from A to B is also common to the two planes.

7. If the extremities of one finite straight line fall upon those of another, two lines are equal. Why?

8. If the paper on which a straight line AB is drawn be so folded that the point A comes upon the point B, and if C be the point on AB through which the *crease* passes, then C is the middle point of AB. Why?

9. When does one angle coincide with another?

10. If the paper on which a straight line AB is drawn be so folded that one portion of the line falls upon another, and if O be the point on AB through which the crease passes, and if C be any other point on the crease, prove that OC is perpendicular to AB.

11. If a straight line starting from the position OA continually rotates about O in the same direction until it comes back to its old position, it turns through four right angles. How?

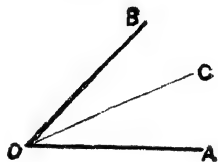
12. AB and CD are two finite straight lines of which AB is greater than CD. Shew how the third postulate can be applied in cutting off from AB a part equal to CD.

13. Shew how the postulates can be applied in drawing from a given point A a straight line equal to a given straight line BC.

14. Two finite straight lines AB and CD are equal to one another. If E be a point on AB and F be a point on CD such that AE is equal to CF, then EB is also equal to FD. Quote the axiom by which this is proved.

15. If one angle ABC be applied to another angle PQR so that B coincides with Q and BC falls upon QR , it is found that BA occupies an intermediate position between QR and QP . Quote the axiom by which it is proved that the angle ABC is less than the angle PQR .

16. AOB is an angle. If the paper be folded so that OA falls on OB and if the straight line OC marks the crease, then OC is the bisector of the angle AOB . How?



SECTION IV.

EXPLANATIONS OF TERMS, SYMBOLS AND ABBREVIATIONS.

1. A complete discussion of a geometrical truth, or of a geometrical construction, is called a **Proposition**.

2. Propositions are of two kinds, *Theorems* and *Problems*

3. A proposition in which a geometrical truth is stated and proved is called a **Theorem**.

4. A proposition in which a geometrical construction is proposed and effected is called a **Problem**.

5. A proposition consists of *four* parts :—The *Enunciation*, the *Particular Enunciation*, the *Construction* and the *Proof*.

(i) The statement in general terms of what is to be proved, or of what is to be done, is called the **Enunciation**.

(ii) The statement of what is to be proved, or of what is to be done, *with special reference to a particular diagram*, is called the **Particular Enunciation**.

(iii) The drawing of such straight lines and circles as may be required to prove the truth of a theorem, or to accomplish the object of a problem, is called the **Construction**.

(iv) The part of a proposition in which, when it is a theorem, the truth of the theorem is proved, or in which

when it is a problem, the construction made is proved to have accomplished the desired end, is called the **Proof**.

Note. In the case of a theorem, the construction may be regarded as forming part of the Proof.

6. The following symbols and abbreviations may be used in writing out propositions :—

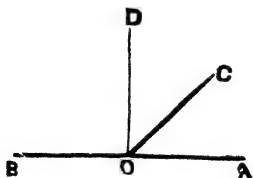
\therefore	for	therefore.	Rectil.	for	Rectilineal.
\because	"	because.	Diff.	"	Defferent.
$=$	"	{ equal to ;	Opp.	"	Opposite.
	"	{ is, or are,	Adj.	"	Adjacent.
	"	{ equal to.	Diag.	"	Diagonal.
\angle	"	angle.	Def.	"	Definition.
rt. \angle	"	right angle.	Post.	"	Postulate.
\triangle	"	triangle.	Ax.	"	Axiom.
\parallel	"	parallel.	Cons.	"	Construction
\perp	"	perpendicular.	Hyp.	"	Hypothesis.
\odot	"	circle.	Equilat.	"	Equilateral.
\bigcirc	"	circumference	Isos.	"	Isosceles.
	"	{ greater than ;	Quadr.	"	Quardrilateral.
$>$	"	{ is or are,	Reqd.	"	Required.
	"	{ greater than.	Int.	"	Interior.
	"	{ less than ;	Ext.	"	Exterior.
$<$	"	{ is, or are,	Prop.	"	Proposition.
	"	{ less than	Th.	"	Theorem.
Pt.	"	Point.	Cor.	"	Corollary.
Str.	"	Straight line	Join AB	"	Draw the
Par ^m .	"	Parallelogram.			straight line
Sq.	"	Square.			from A to B.
Rect.	"	Rectangle.			

SECTION V.

THEOREMS.

Theorem 1. (EUCLID I. 13.)

If two adjacent angles be such that their non-coincident arms are in the same straight line, then these two angles are together equal to two right angles.



Let AOC and COB be two adjacent angles, of which the non-coincident arms OA and OB are in the same straight line.

To prove that the angles AOC and COB are together equal to two right angles.

Cons. Suppose OD is drawn perpendicular to the straight line AOB.

Proof. Then the \angle s AOC, COB are together
= the three \angle s AOC, COD, DOB.

Also the \angle s AOD, DOB are together
= the same three \angle s AOC, COD, DOB.

\therefore the \angle s AOC, COB are together
= the \angle s AOD, DOB (Ax. 1.)
= two right angles. Q. E. D.

. **Alternative proof.**

The \angle s **AOC**, **COB** are together
 = the *straight angle* **AOB**
 = two right angles. (*Def. 9, Note 3.*)

Q. E. D.

Corollary 1. *If two straight lines cut one another, the four angles thus formed are together equal to four right angles.*

Cor. 2. *If any number of straight lines diverge from a given point, the sum of the consecutive angles so formed is equal to four right angles.*

For, if any one of the lines be produced through the common point, the sum of the angles on *each* side of this line = two rt. \angle s; and \therefore the sum of the \angle s on both sides = four right angles.

Note 1. The letters Q. E. D., which are found at the end of a theorem stand for *Quod erat demonstrandum*, which was to be proved.

Note 2. A geometrical truth which can be easily deduced from a theorem is called a **Corollary** to it.

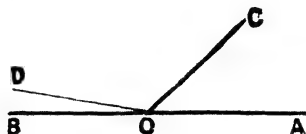
Note 3. In the enunciation of a theorem, that which is *assumed to be true* is called the **Hypothesis**, and that which is *required to be proved* is called the **Conclusion**. In Theorem 1, the *hypothesis* is that the arms **OA** and **OB**, of the adjacent angles **AOC** and **COB**, are in the same straight line; and the *conclusion* is that the angles **AOC** and **COB** are together equal to two right angles.

Note 4. When two angles are together equal to two right angles, each of them is called the **Supplement** of the other, and the two angles are said to be **Supplementary** to each other. Thus, in the diagram of Theorem 1, each of the angles **AOC** and **COB** is the *supplement* of the other.

Note 5. When two angles are together equal to one right angle, each of them is called the **complement** of the other, and the two angles are said to be **complementary** to each other. Thus, in the diagram of Theorem 1, each of the angles **AOC** and **COD** is the *complement* of the other.

Theorem 2 (Euc. I. 14.)

If two adjacent angles be such that they are together equal to two right angles, then their non-coincident arms are in the same straight line.



Let the adjacent angles AOC and COB be together equal to two right angles.

To prove that OB is in the same str. line with AO .

Proof. OB must be in one of two conditions; it must be either in the same str. line with AO , or not so.

Suppose OB , is *not* in the same str. line with AO , and that OD , which is different from OB , is the produced part of AO .

Now, since OA and OD are in the same str. line, therefore the \angle s AOC and COD are together = two right angles.

But, by hypothesis, the \angle s AOC and COB are together = two right angles.

\therefore the $\angle \text{AOC} +$ the $\angle \text{COD} =$ the $\angle \text{AOC} +$ the $\angle \text{COB}$, (Ax. 1.)

\therefore the $\angle \text{COD} =$ the $\angle \text{COB}$; (Ax. 3.)

i.e., a part is equal to the whole, which is *impossible*.

(Ax. 9.)

Thus, it is *absurd* to suppose that **OB** is *not* in the same straight line with **AO**.

Clearly therefore **OB** is in the same str. line with **AO**.

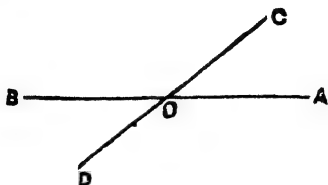
Q. E. D.

Note 1. In Theorem 2, the *hypothesis* is that *the angles AOC and COB are together equal to two right angles*; and the *conclusion* is that *the arms, OA and OB, are in the same straight line*. Clearly therefore the hypothesis and conclusion of Theorem 2 are respectively the conclusion and hypothesis of Theorem 1. From such a relation between the two theorems, Theorem 2 is said to be the converse of Theorem 1. In other words, if two theorems be so related that *the hypothesis of each is the conclusion of the other*, then either of them is said to be the **converse** of the other.

Note 2. The method of proof used in this proposition is called the **Indirect Method**. It is quite clear that the straight line **OB** must *either be* in the same straight line with **AO**, *or not*; hence when the second supposition is proved to be *absurd*, the inevitable conclusion is that the first is correct. Thus, the *indirect method* consists in proving the truth of a theorem *by proving the absurdity of the supposition that the theorem is not true*. This method is also called **reductio ad absurdum**; it is generally employed in proving the converse of an established theorem.

Theorem 3. (Euc. I. 15.)

If two straight lines cut one another, the vertically opposite angles are equal.



Let the straight lines AB, CD cut one another at the point O.

To prove that

$$\begin{array}{l} \text{(1) the } \angle AOC = \text{the } \angle BOD \} \\ \text{and (2) the } \angle COB = \text{the } \angle DOA \} \end{array}$$

Proof. The $\angle AOC + \text{the } \angle COB = \text{two rt. angles}$ }
 also, the $\angle COB + \text{the } \angle BOD = \text{two rt. angles}$; }
(Th. 1.)

$$\begin{array}{l} \therefore \text{ the } \angle AOC + \text{the } \angle COB \\ \qquad \qquad \qquad = \text{the } \angle COB + \text{the } \angle BOD. \end{array} \quad (Ax. 1.)$$

$$\therefore \text{ the } \angle AOC = \text{the } \angle BOD \quad (Ax. 2.)$$

$$\begin{array}{l} \text{by a similar mode of reasoning, the } \angle COB \\ \qquad \qquad \qquad = \text{the } \angle DOA. \end{array}$$

Q. E. D.

Cor. *Supplements of the same angle are equal to one another.*

EXERCISE (2).

1. If two straight lines CC, OD diverge on opposite sides of the straight line AOB, so that the $\angle AOC$

=the $\angle BOD$, prove that OD is in the same straight line with CO .

2. Two straight lines AB , CD cut one another at O . Prove that the bisector of the $\angle AOC$, when produced through O , also bisects the $\angle BOD$.

3. ABC and PQR are two acute \angle s equal to one another. Prove that their *supplements* are equal. Prove also that their *complements* are equal.

4. If two straight lines cut one another and if one of the four angles so formed be a right angle, prove that each of the other three angles is a right angle.

5. If four straight lines OA , OB , OC , OD diverge from a point O , so that each of the \angle s AOB , BOC , COD , DOA is a right angle, prove that AO , OC are in the same straight line, and so are BO , OD .

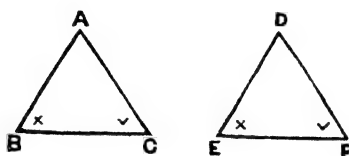
6. Prove that the bisectors of two vertically opposite angles are in one and the same straight line.

7. If four straight lines OA , OB , OC , OD diverge from a point O , so that the $\angle AOB = \text{the } \angle COD$ and the $\angle BOC = \text{the } \angle DOA$, prove that AO , OC are in the same straight line, and so are BO , OD .

8. Prove that the internal and external bisectors of an angle are at right angles to each other. (The bisector of an angle and that of its supplement are called the *internal and external bisectors* of the angle.)

Theorem 4. (Euc. I. 4.)

If two triangles be such that two sides and the included angle of one are respectively equal to two sides and the included angle of the other, then these two triangles are equal in all respects.



Let $\triangle ABC$, $\triangle DEF$ be two triangles such that

$$(1) \quad AB = DE,$$

$$(2) \quad AC = DF,$$

and (3) the included $\angle BAC =$ the included $\angle EDF$. }

To prove that the $\triangle ABC =$ the $\triangle DEF$ in all respects.

Proof. Apply the $\triangle ABC$ to the $\triangle DEF$, so that A falls on D and AB on DE . Then, because the $\angle BAC =$ the $\angle EDF$. (Hyp.)

$\therefore AC$ must fall upon DF .

Now, since AB falls on DE , and $AB = DE$,

$\therefore B$ falls on E ;

and, since AC falls on DF , and $AC = DF$,

$\therefore C$ falls on F .

$\therefore BC$ coincides with EF . }

(Ax. 10.)

Thus, the sides AB , AC , BC coincide respectively with DE , DF , EF ;

and \therefore the $\triangle ABC$ coincides with the $\triangle DEF$.

Hence the $\triangle ABC =$ the $\triangle DEF$ *in all respects*.

(Ax. 2.)

Q. E. D.

Note 1. Every triangle is said to have six *parts*, the three sides and the three angles. The sides of a triangle also enclose an *amount of surface* which is called its *area*.

Note 2. When one triangle coincides with another, it is evident that all the six parts and the area of the one are respectively equal to all the six parts and the area of the other.

Note 3. In Theorem 4, it is shewn that under the conditions given, the triangles can be made to *coincide* with one another, and hence they are equal *in all respects* i.e., all the six parts and the area of the one are respectively equal to all the six parts and the area of the other.

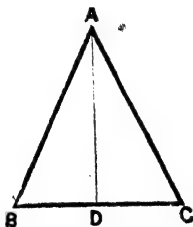
Note 4. When two triangles are equal in all respects, they are said to be **identically equal** or **congruent**.

Note 5. In *congruent* triangles, *angles opposite to equal sides are equal*, as is evident from the above diagram.

Note 6. When we say that the triangles are equal *in all respects*, this certainly includes the three equalities which form the *hypothesis*— $AB=DE$, $AC=DF$, $\angle A=\angle D$. Hence, to make a clear distinction between hypothesis and conclusion, we should say that the *conclusion* is that $BC=EF$, $\angle B=\angle E$, $\angle C=\angle F$, and the area of $\triangle ABC =$ the area of $\triangle DEF$. That is, of the seven equalities possible, if the above three hold good, the other four must *necessarily* follow.

Theorem 5. (EUC. I. 5.)

If two sides of a triangle are equal, then also the angles opposite to these two sides are equal.



Let ABC be a \triangle , having the side $AC =$ the side AB .

To prove that the $\angle ABC =$ the $\angle ACB$.

Proof. Let AD be the bisector of the $\angle BAC$, and let it meet BC in D . Fold the paper about AD from left to right.

Then since the $\angle BAD =$ the $\angle CAD$,

$\therefore AB$ must fall upon AC .

And, since $AB = AC$, (Hyp.)

$\therefore B$ must fall on C , and consequently BD will coincide with DC .

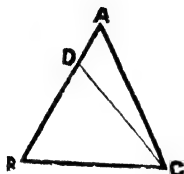
Thus the $\angle ABD$ coincides with the $\angle ACD$,
and is \therefore equal to it. (Ax. 8.)

Q. E. D.

Cor. *If a triangle is equilateral, it is also equiangular.*

Theorem 6. (EUC. I. 6.)

If two angles of a triangle are equal, then also the sides opposite to these two angles are equal.



Let ABC be a \triangle , having the $\angle ABC = \text{the } \angle ACB$.

To prove that the side $AC = \text{the side } AB$.

Proof. Suppose AC is not equal to AB . Then one of them must be the greater; let $AB > AC$, and from BA cut off $BD = AC$.

Join DC .

Then in the \triangle s DBC, ACB ,

because (1) $DB = AC$,

(2) BC is common to both,

and (3) the included $\angle DBC$

$= \text{the included } \angle ACB$; (Hyp.)

\therefore the $\triangle DBC = \text{the } \triangle ACB$; (Th. 4.)

i.e., a part is equal to the whole,

which is impossible. (Ax. 9.)

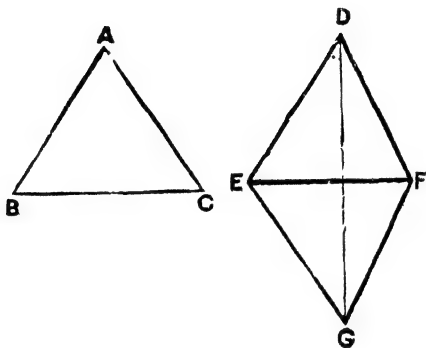
Thus it is *absurd* to suppose that AC is *not* equal to AB . Undoubtedly $\therefore AC = AB$. Q. E. D.

Cor. *If a triangle is equiangular, it is also equilateral.*

Note. Theorem 6 is evidently the *converse* of Theorem 5.

Theorem 7. (Euc. I. 8.)

If two triangles be such that the three sides of one are respectively equal to the three sides of the other, then these two triangles are equal in all respects.



Let ABC and DEF be two \triangle s. having

$$\left. \begin{array}{l} AB = DE, \\ AC = DF, \\ BC = EF. \end{array} \right\}$$

and

To prove that the two \triangle s are equal in all respects.

Proof. Of the sides of the $\triangle DEF$, let EF be that which is *not less* than either of the other two.

Apply the $\triangle ABC$ to the $\triangle DEF$, so that BC , which $= EF$, may coincide with EF , and so that A may fall on that side of EF which is remote from D .

Let GEF be the new position of the $\triangle ABC$.

Join DG .

Then since $EG = ED$, (Hyp.)

\therefore the $\angle EDG =$ the $\angle EGD$; (Th. 5.)

and since $FG = FD$, (Hyp.)

\therefore the $\angle FDG =$ the $\angle FGD$. (Th. 5.)

Hence, the whole $\angle EDF =$ the whole $\angle EGF$;
(Ax. 2.)

i.e., the $\angle EDF =$ the $\angle BAC$.

Now in the \triangle s ABC , DEF , we have

(1) $AB = DE$,

(2) $AC = DF$,

and (3) the included $\angle BAC =$ the included $\angle EDF$;

\therefore the $\triangle ABC =$ the $\triangle DEF$ in all respects.
(Th. 4.)

Q. E. D.

Note. In Theorem 7, it is proved that in the two \triangle s ABC and DEF , if it be *given* that $AB = DE$, $AC = DF$, and $BC = EF$, then it *necessarily* follows that $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$, and the area of the $\triangle ABC =$ the area of the $\triangle DEF$. It must be remembered that *those angles are equal which are opposite to equal sides*.

EXERCISE (3).

1. Prove that the bisector of the vertical angle of an isosceles triangle bisects the base and is also perpendicular to it.

2. Prove that the straight line drawn from the vertex of an isosceles triangle to the mid-point of the base bisects the vertical angle and is also perpendicular to the base.

3. If BC be the base of an isosceles triangle ABC , and if AB , AC be produced, prove that the angles formed on the other side of the base are equal.

4. AB is a given str. line ; if the str. line CD passes through the mid-point of AB and is also perpendicular to it, prove that *any* point in CD is equidistant from the points A and B .

5. ABC and DBC are two isos. Δ s on the same base BC and on the same side of it. Prove that if AD be joined, and produced to meet the base, it will bisect the vertical angles of the Δ s and be also perpendicular to the base.

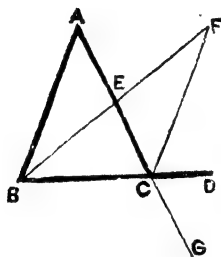
6. ABC is an isos. Δ on the base BC ; if the bisectors of the base angles meet at D , and if AD be joined, prove that AD bisects the $\angle BAC$.

7. Prove that the angles opposite to the unequal sides of a triangle are also unequal.

8. If two quadrilaterals $ABCD$ and $EFGH$ be such that $AB=EF$, $BC=FG$, $CD=GH$, $\angle ABC = \angle EFG$, and $\angle BCD = \angle FGH$, shew that the two figures can be made to coincide with one another.

Theorem 8. (Euc. I. 16.)

If one side of a triangle be produced, then the exterior angle is greater than either of the interior opposite angles



Let one side BC of the $\triangle ABC$ be produced to D .

To prove that the exterior $\angle ACD$ is greater than either of the interior opposite \angle s BAC, ABC .

Let E be the middle point of AC .

Join BE ; and produce it to F , making $EF = BE$.

Join CF .

Proof. In the \triangle s AEB, CEF , we have

$$(1) \quad AE = CE,$$

$$(2) \quad EB = EF,$$

$$\text{and} \quad (3) \quad \angle AEB = \text{the vertically opp. } \angle CEF, \quad (\text{Th. 3.})$$

$$\therefore \text{ the two } \triangle\text{s are congruent.} \quad (\text{Th. 4.})$$

Hence, the $\angle BAE =$ the $\angle ECF$, being opp. to the equal sides BE, EF .

But the $\angle ECD$ is greater than the $\angle ECF$;

\therefore the $\angle ECD$ is greater than the $\angle BAE$;

i.e., the $\angle ACD$ is $>$ the $\angle BAC$.

In the same way, producing AC to G and joining A to the middle point of BC , it may be proved that the $\angle BCG$ is $>$ the $\angle ABC$.

But the $\angle BCG =$ the vertically opp. $\angle ACD$;

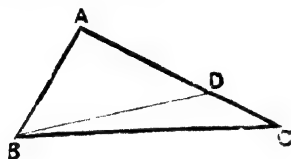
\therefore the $\angle ACD$ is $>$ the $\angle ABC$. Q. E. D.

Cor. *Any two angles of a triangle are together less than two right angles.*

For, in the above diagram, the $\angle ABC$ is less than the $\angle ACD$; \therefore the $\angle ABC +$ the $\angle ACB$ is less than the $\angle ACD +$ the $\angle ACB$, and is $\therefore <$ two rt \angle s.

Theorem 9. (EUC. I. 18.)

If one side of a triangle be greater than another, then the angle opposite to the greater side is greater than the angle opposite to the less.



Let ABC be a \triangle , having the side $AC >$ the side AB .

To prove that the $\angle ABC$ is $>$ the $\angle ACB$.

From AC cut off $AD = AB$.

Join BD .

Proof. Because $AB = AD$,

\therefore the $\angle ADB =$ the $\angle ABD$. (Th. 5.)

But the ext. $\angle ADB$ of the $\triangle BDC$ is greater than the int. opp. $\angle DCB$, i.e., greater than the $\angle ACB$.

\therefore the $\angle ABD$ is $>$ the $\angle ACB$.

Much more then is the $\angle ABC$ greater than the $\angle ACB$.

Q. E. D.

Theorem 10. (Euc. I. 19.)

If one angle of a triangle be greater than another, then the side opposite to the greater angle is greater than the side opposite to the less..



Let ABC be a \triangle , having the $\angle ABC$ greater than the $\angle ACB$.

To prove that the side $AC >$ the side AB .

Proof. If AC is not greater than AB , then it must be *either* $= AB$, *or* less than AB .

(1) If $AC = AB$,

then the $\angle ABC =$ the $\angle ACB$; (Th. 5.)
which is impossible, by hypothesis.

(2) If $AB < AC$.

then the $\angle ABC$ is less than the $\angle ACB$, (Th. 9.)
which is also impossible, by hypothesis.

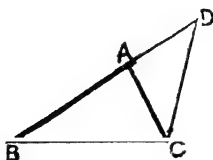
Thus, AC is neither equal to, nor less than AB ;

$\therefore AC > AB$.

Q. E. D.

Theorem 11. (Euc. I. 20.)

Any two sides of a triangle are together greater than the third side.



Let ABC be a triangle.

To prove that any two of its sides, AB and AC, are together greater than the third side BC.

Produce BA to D, making $AD = AC$.

Join DC.

Proof. Because $AD = AC$.

\therefore the $\angle ACD = \angle ADC$. (Th. 5.)

But the $\angle BCD$ is $>$ the $\angle ACD$;

\therefore the $\angle BCD$ is also $>$ the $\angle ADC$,
that is, the $\angle BCD$ is $>$ the $\angle BDC$.

Hence, from the $\triangle BDC$, we have
BD greater than BC. (Th. 10.)

But BA and AC are together = BD;

\therefore BA and AC are together $>$ BC.

Q. E. D.

Cor. *The difference of any two sides of a triangle is less than the third side.*

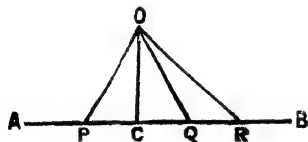
For, $AB - AC$ is $<$ BC, if $AB < AC + BC$, and this latter is true.

Note. A straight line may be regarded as *the shortest distance between its extreme points*. From this point of view, the above theorem is self-evident.

Theorem 12.

Of all the straight lines that can be drawn to a given straight line from a given point outside it,

- (i) *the perpendicular is the shortest ;*
- (ii) *every two that meet the straight line at equal distances from the foot of the perpendicular, are equal ;*
- (iii) *one that meets the straight line at a greater distance from the foot of the perpendicular than another, is greater than that other.*



Let OC be the perpendicular from the given point O to the given str. line AB.

- (i) Let OP be *any other* str. line from O meeting the given str. line at P.

To prove that $OP > OC$.

Proof. Because the side PC of the $\triangle OCP$ is produced to B,

$$\therefore \text{the } \angle OCB > \text{the } \angle OPC. \quad (\text{Th. 8.})$$

But the $\angle OCB = \text{the } \angle OCP$,

$$\therefore \text{the } \angle OCP \text{ is } > \text{the } \angle OPC.$$

Hence $OP > OC$. (Th. 10.)

- (ii) Let the two str. lines OP and OQ, drawn from the point O to the given str. line, be such that $CQ = CP$.

To prove that $OQ = OP$.

Proof. In the \triangle s OCP and OCQ , we have

(1) $CP = CQ$,

(2) CO common,

and (3) the $\angle OCP =$ the $\angle OCQ$;

\therefore the two triangles are congruent. (Th. 4.)

Hence $OQ = OP$.

(iii) Let the two str. lines OQ and OR , drawn from the point O to the given str. lines, be such that $CR > CQ$.

To prove that $OR > OQ$.

Proof. The ext. $\angle OQR$ is $>$ the int. opp. $\angle OCQ$:
(Th. 8.)

\therefore the $\angle OQR$ is *obtuse*.

Also the ext. $\angle OCA$ is $>$ the int. opp. $\angle ORC$; (Th. 8.)

\therefore the $\angle ORC$ is *acute*.

Hence the $\angle OQR$ is $>$ the $\angle ORQ$:

$\therefore OR > OQ$. (Th. 10.)
Q. E. D.

Note. The length of the perpendicular from a given point upon a given straight line is called the **distance** of the point from that line.

EXERCISE (4).

1. ABC is a \triangle ; AD bisects the $\angle BAC$ and meets BC at D . Prove that $AB > BD$, and $CA > CD$. Hence shew that $AB + AC > BC$.

2. The two base \angle s of any \triangle are together less than two right angles. Prove this by joining the vertex to any point of the base.

3. Prove that in a triangle one angle of which is either right or obtuse, *each* of the other two angles is *acute*.

4. If in a $\triangle ABC$, the $\angle BAC$ be greater than the $\angle ABC$, prove that the angle ABC is acute.

5. Prove that only one perpendicular can be drawn to a given straight line from a given point outside it

6. Prove that from a given point outside a given straight line there can be drawn to it two and *only two* straight lines that are equal to one another.

7. D is any point within a $\triangle ABC$; if BD and CD are joined, prove that the angle BDC is greater than the angle BAC .

8. D is any point within a $\triangle ABC$; if BD and CD are joined, prove that $AB + AC > BD + DC$.

9. If O is any point within a $\triangle ABC$, prove that the sum of the lines OA, OB, OC is less than the sum, but greater than half the sum, of the sides of the triangle.

10. If $ABCD$ be a quadrilateral such that AB is the greatest and CD is the least of its sides, prove that the $\angle BCD$ is greater than the $\angle BAD$, and the $\angle ADC$ is greater than the $\angle ABC$.

11. Prove that the sum of the diagonals of any quadrilateral is less than the sum, but greater than half the sum, of the sides of the quadrilateral.

12. AB, AD are two equal sides of a $\triangle ABD$; if AD is produced to any point C and BC is joined, prove that $DC < BC$. Hence deduce a construction for proving that the difference of any two sides of a triangle is less than the third side.

Theorem 13. (Euc. I. 27 and 28.)

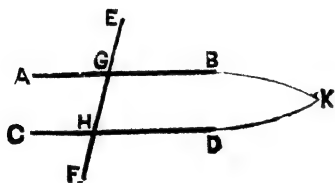
If a straight line, cutting two other straight lines makes,

(i) the alternate angles equal;

or (ii) an exterior angle equal to the interior opposite angle on the same side of it;

or (iii) the interior angles on the same side equal to two right angles;

then those two straight lines are parallel.



Let the str. line **EF** cut the two str. lines **AB**, **CD** at **G** and **H**.

(i) Let the alternate \angle s **AGH**, **GHD** be equal to one another.

To prove that **AB and **CD** are parallel.**

proof. If **AB** and **CD** are *not* parallel, they will meet when produced either towards **B** and **D**, or towards **A** and **C**.

Suppose, when produced towards **B** and **D**, they meet at the point **K**.

Then **KGH** is a Δ , of which the side **KG** has been produced to **A**;

\therefore the ext. \angle AGH is $>$ the int. opp. \angle GHK.

(Th. 8.)

But this is contrary to the hypothesis ;

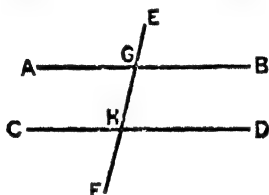
\therefore AB and CD cannot meet when produced towards B and D.

Similarly they can neither meet towards A and C.

Hence AB and CD are parallel.

(ii) Let the ext. \angle EGB be = the int. opp. \angle GHD.

To prove that AB and CD are parallel.



Proof. Because the \angle EGB = the \angle GHD,

and the \angle EGB = the vertically opp. \angle AGH ;

\therefore the \angle AGH = the \angle GHD, (Ax. 1.)

which are *alternate* angles.

\therefore AB and CD are parallel.

(iii) Let the two int. \angle s BGH, GHD, on the same side of EF, be together = two rt. \angle s.

To prove that AB and CD are parallel.

Proof. Because the \angle BGH + the \angle GHD
= two rt. \angle s,

and also the \angle BGH + the \angle AGH

= two rt. \angle s ; (Th. 1.)

\therefore the \angle BGH + the \angle GHD

= the \angle BGH + the \angle AGH. (Ax. 1.)

Hence, the $\angle \text{GHD} = \text{the } \angle \text{AGH}$, (Ax. 3.)
which are *alternate* angles.

\therefore AB and CD are parallel. Q. E. D.

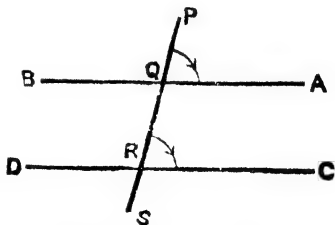
Cor. If two straight lines are perpendicular to the same straight line, they are parallel to one another.

Note 1. Altogether eight \angle s have been formed at the points G and H. Of these \angle s, the two that lie above the line AB, as well as the two that lie below CD, are called *exterior* angles; whilst the remaining four \angle s are called *interior* angles. They are only two pairs of *alternate* \angle s; \angle s AGH and GHD forming one pair, and the \angle s BGH and GHC forming the other. Any ext. \angle and the int. opp. \angle on the same side of EF are said to form a pair of *corresponding* \angle s; thus the \angle s EGB and GHD are a pair of *corresponding* \angle s; similarly there are three other pairs of corresponding \angle s.

Note 2. There are three different hypothesis, and in each case the conclusion is one and the same. Hence the proposition might very well be split up into three separate propositions.

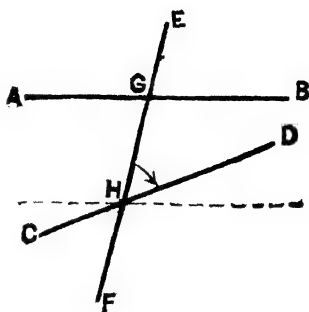
Note 3. Any straight line that cuts two or more given lines is called a *transversal*. Thus, in the above diagram, the straight line EF is a transversal.

Note 4. If any fixed and unlimited str. line PQRS cuts any two str. lines AB, CD at Q and R, AB is \parallel to CD when the \angle s AQP and CRP are *equal*. Hence if AB and CD be both supposed to be initially on the fixed line PS,



and then to turn in the *watch-hand* way, AB about the point Q and CD about the point R, they will become parallel when they have undergone the *same amount of turning*. Hence, two str. lines are parallel when they have the *same direction*.

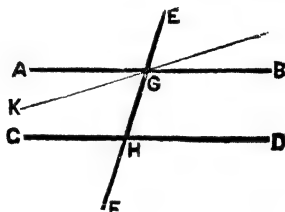
Note 5. Let a str. line **EF** cut a str. line **AB** at **G**. Let another str. line **CD** rotate about the pt. **H** on **EF**, in the direction of the arrow head, starting from a position of coincidence with **EF**. Then it is evident that the $\angle \text{EHD}$ gradually increases, whilst the $\angle \text{DHF}$ gradually decreases. Hence there is one position of **CD** and *one only* in which the $\angle \text{EHD} = \text{the } \angle \text{EGB}$, and in this position **CD** is parallel to **AB**.



Theorem 14. (Euc. I. 29.)

If a straight line cuts two parallel straight lines, it makes,

- (i) the alternate angles equal to one another ;*
- (ii) the exterior angle equal to the interior opposite angle on the same side of it ;*
- and (iii) the two interior angles on the same side together equal to two right angles.*



Let the str. line EF cut the \parallel str. lines AB , CD at the pts. G and H .

(1) To prove that the $\angle AGH =$ the alt. $\angle GHD$.

Proof. If the $\angle AGH$ be not equal to the $\angle GHD$, let the $\angle KGH$ be equal to it.

Now since the \angle s KGH and GHD are alt. \angle s, and equal to one another.

\therefore KG is parallel to CD . (Th. 13.)

But, by hypothesis, AB is parallel to CD ;

\therefore the two str. lines AG , KG , which intersect one another, are both \parallel to CD , which is impossible.

(Playfair's Axiom.)

Thus, the $\angle AGH$ cannot be unequal to the $\angle GHD$;

$\therefore \angle AGH =$ the $\angle GHD$.

(2) To prove that the ext. $\angle EGB$
 $=$ the int. opp. $\angle GHD$.

Proof. Because the $\angle AGH$

= the $\angle GHD$, (*already proved.*)

and also the $\angle AGH$ = the vertically opp. $\angle EGB$;

\therefore the $\angle EGB$ = the $\angle GHD$. (*Ax. 1.*)

(3) To prove that the two interior \angle s BGH , GHD are together = two rt. \angle s.

Proof. Because the $\angle EGB$ = the $\angle GHD$,

(*proved above.*)

\therefore the $\angle EGB$ + the $\angle BGH$ = the $\angle GHD$ + the $\angle BGH$.

But the \angle s EGB , BGH are together

= two rt. \angle s ;

(*Th. 1.*)

\therefore the \angle s BGH , GHD are together

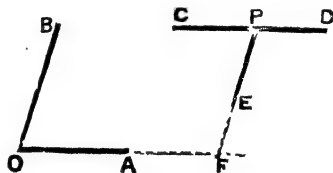
= two rt. \angle s.

Q. E. D.

Cor. 1. If a straight line is perpendicular to one of two parallel straight lines, it is also perpendicular to the other.

Cor. 2. Two angles having parallel arms are either equal or supplementary.

In the accompanying diagram, let the str. line CPD be \parallel to OA and PE to OB . If OA and PE be produced to meet at F , it is easy to see that \angle s

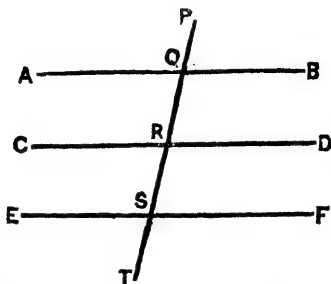


AOB , CPE are *equal*, each of them being supplementary to the $\angle PFO$; and that \angle s AOB , DPE are *supplementary*, the $\angle DPE$ being = the $\angle PFO$ which is supplementary to the $\angle AOB$.

Note. In any str. line AB , the direction from A to B is *opposite* to that from B to A . This idea is also expressed by saying that the line AB is opposite in **sense** to the line BA .

Theorem 15. (Euc. I. 30.)

Straight lines which are parallel to the same straight line are parallel to one another.



Let the straight lines AB, CD be each parallel to EF.

To prove that AB is parallel to CD

Draw any straight line PT cutting AB, CD and EF in the points Q, R and S.

Proof. Since AB is \parallel to EF, and PT cuts them,

\therefore the $\angle AQS =$ the alt. $\angle QSF$. (Th. 14.)

Again, since CD is \parallel to EF, and PT cuts them,

\therefore the ext. $\angle QRD =$ the int. opp. $\angle QSF$. (Th. 14.)

Hence the $\angle AQR =$ the $\angle QRD$. (Ax. 1.)

But these are alternate \angle s ;

\therefore AB is \parallel to CD. Q. E. D.

Alternative Proof. If AB is not \parallel to CD, they will meet when produced ; and then we have two intersecting str. lines both parallel to a third straight line, which is impossible. (Playfair's Axiom.)

Hence AB and CD cannot meet, and \therefore they are parallel.

Note. If EF lies between AB and CD, the truth of the theorem becomes obvious. For, if AB and CD were to meet, one of them would cut EF, which is impossible. Hence AB and CD cannot meet, and \therefore they are parallel.

EXERCISE (5).

1. If a str. line cuts two parallel str. lines prove that the bisectors of any pair of alt. angles are parallel.

2. If one angle of a parallelogram be a right angle, prove that the other three angles also are right angles.

3. If a straight line intersecting two other straight lines make two exterior angles on the same side of the line together equal to two right angles, prove that the two straight lines are parallel.

4. If through the vertex of an isosceles triangle a straight line be drawn parallel to the base, prove that it bisects the exterior vertical angle.

5. If the side BC of a triangle ABC be produced to D, shew that the bisectors of the angles BAC, ACD cannot be parallel.

6. If two straight lines be such that one of them is perpendicular to a given straight line while the other is not, prove that they must intersect each other.

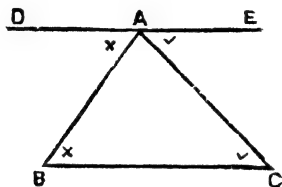
7. If any two straight lines be drawn perpendicular to two given intersecting straight lines, prove that they must intersect.

8. Through each angular point of a triangle a str. line is drawn parallel to the opposite side ; prove that the triangle formed by these three str. lines is equiangular to the given triangle.

9. If a quadrilateral ABCD be such that the diagonals AC, BD bisect each other, prove that the quadrilateral is a parallelogram.

Theorem 16. (Euc. I. 32.)

The sum of the angles of a triangle is equal to two right angles.



Let ABC be a triangle.

To prove that the \angle s ABC , BCA , CAB are together = two rt. \angle s.

Suppose, through the pt. A , the str. line DAE has been drawn \parallel to BC .

Proof. Since DE is \parallel to BC and AB meets them

\therefore the $\angle ABC$ = the alt. $\angle DAB$.

Again, since DE is \parallel to BC and AC meets them,

\therefore the $\angle BCA$ = the alt. $\angle CAE$.

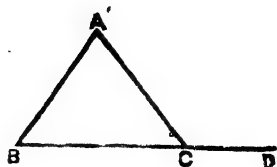
Hence, the \angle s ABC , BCA are together
= the \angle s DAB , CAE .

To each of these equals add the $\angle BAC$;

\therefore the sum of the \angle s ABC , BCA , CAB
= the sum of the \angle s DAB , BAC , CAE
= two right angles. Q. E. D.

Cor. 1. *If any side of a triangle is produced, the exterior angle is equal to the sum of the two interior opposite angles.*

In the accompanying diagram, the sum of the \angle s BCA , ACD = two rt. \angle s = the $\angle BCA$ + the $\angle CAB$ + the $\angle ABC$; \therefore the $\angle ACD$ = the $\angle CAB$ + the $\angle ABC$. (*Ax. 3.*)



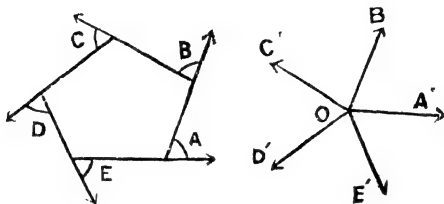
Cor. 2. *If two triangles have two angles of the one respectively equal to two angles of the other, then the third angle of the one is equal to third angle of the other.*

Note 1. If a right angle be supposed to be divided into 90 equal parts, each part is called a **degree**; a sixtieth part of a degree is called a **minute**, and a sixtieth part of a minute is called a **second**. In other words, 60 seconds make a minute, 60 minutes make a degree, and 90 degrees make a right angle. 8 degrees, 13 minutes and 20 seconds is written as $8^{\circ} 13' 20''$.

Note 2. The sum of the angles of a triangle = 180° .

Theorem 17.

If the sides of a rectilineal figure, which has no re-entrant angle, are produced in order, then the sum of the exterior angles so formed is equal to four right angles.



Let the sides of the rectilineal figure $ABCDE$ be respectively produced in the directions shewn by the arrow-heads.

To prove that the exterior \angle s A, B, C, D, E are together = four right \angle s.

Take any point O , and suppose the str. lines OA, OB', OC', OD', OE' are drawn \parallel respectively to the sides EA, AB, BC, CD, DE ; each being drawn in the *sense* in which the corresponding side is produced.

Proof. Since OA' is \parallel to EA produced, and OB' is \parallel to AB ,

\therefore the ext. $\angle A =$ the $\angle A'OB'$. (*Th. 14, Cor. 3.*)

Similarly, the ext. $\angle B =$ the $\angle B'OC'$,
the ext. $\angle C =$ the $\angle C'OD'$,
the ext. $\angle D =$ the $\angle D'OE'$.
the ext. $\angle E =$ the $\angle E'O'A'$.

Hence, the sum of the ext. \angle s A, B, C, D, E

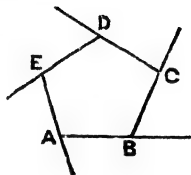
= the sum of the \angle s at O.

= four right angles. (Th. 1, Cor. 2.)

Q. E. D.

Cor. If a convex* rectilineal figure has n sides, then the sum of all its interior angles is equal to $(2n-4)$ right angles.

Let ABCDE be a convex rectilineal figure having n sides; then it has also n interior \angle s and n vertices.



If the sides be produced in order, we find that the sum of the int. and ext. \angle s at each vertex = 2 rt. \angle s.

Hence, the sum of the int. and ext. \angle s at the n vertices = $2n$ rt. \angle s.

That is, the sum of all the int. \angle s + the sum of the ext. \angle s = $2n$ rt. \angle s.

But the sum of the ext. \angle s = 4 rt. \angle s; \therefore the sum of the int. \angle s = $(2n-4)$ rt. \angle s.

Note 1. The sides of a rectilineal figure are said to be produced *in order*, when, considering a point to move round the figure along its sides, each side is produced in the *sense* in which the point is moving along it.

Note 2. A rectilineal figure is said to be **regular** when it is both equilateral and equiangular. A rectilineal figure having five sides is called a **pentagon**, and one having six sides is called a **hexagon**. Hence a regular pentagon, or a regular hexagon, is one which has all its sides and angles equal.

Note 3. The sum of the interior angles of a pentagon = 6 right angles. Hence each angle of a *regular* pentagon = one-fifth of 6 rt. angles = 108° . Similarly each angle of a *regular* hexagon = 120° .

*A rectilineal figure is said to be *convex* when it has no re-entrant angle.

EXERCISE (6).

1. Prove that in a rt. angled triangle, each of the acute angles is the complement of the other.

2. If one angle of a triangle be equal to the sum of the other two, prove that the triangle is rt. angled.

3. If one angle of a triangle be greater than the sum of the other two, prove that the triangle is obtuse-angled.

4. Prove that the sum of the angles of a quadrilateral is equal to four right angles.

5. If one side of a triangle be produced the exterior angle is equal to the sum of the two interior opposite angles. Shew how this can be proved without deducing it as a corollary to Theorem 16.

6. Prove by the application of Theorem 16, that the interior angles of a pentagon are together equal to six right angles.

7. Find the magnitude of an angle of a regular polygon of ten sides.

8. Prove that the acute angle between two straight lines is equal to the acute angle between any two straight lines at right angles to them.

9 Find the magnitude of the vertical angle of an isosceles triangle each of the base angles of which is double of the vertical angle.

10. Draw a diagram shewing that a heptagon (*i.e.*, a rectilineal figure of seven sides) can be divided into five

triangles, and hence shew that the sum of its interior angles is equal to ten right angles.

11. Prove that the straight line bisecting the exterior vertical angle of an isosceles triangle is parallel to the base.

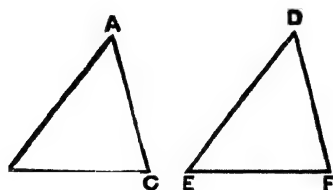
12. ABC , ABD , ACE are equilateral triangles ; prove that the points D , A , E are in the same straight line.

13. A is the vertex of an isosceles triangle ABC , and BA is produced to D , so that AD is equal to AB ; if DC is joined, prove that BCD is a right angle.

14. BC is the hypotenuse of a right-angled triangle ABC . If AD be drawn to meet BC , so that the angle BAD is equal to the angle ABD , prove that D is the middle point of BC and that AD is half of BC .

Theorem 18. (Euc. I. 26.)

If two triangles be such that two angles and a side of the one triangle are respectively equal to two angles and the corresponding side of the other, then these two triangles are equal in every respect.



Let the \angle s ABC , BCA of the $\triangle ABC$ be respectively equal to the \angle s DEF , EFD of the $\triangle DEF$. Also

- (1) let the side BC be = the *corresponding* side EF ,
 or (2) let the side AB be = the *corresponding* side DE ,
 or (3) let the side AC be = the *corresponding* side DF .

To prove that in each of these three cases the two \triangle s are congruent.

Case I. Apply the $\triangle ABC$ to the $\triangle DEF$ so that B may be on E and BC on EF .

Then, because $BC = EF$,

$\therefore C$ must coincide with F .

Now, since the $\angle B =$ the $\angle E$,

$\therefore BA$ must fall on ED ;

and since the $\angle C =$ the $\angle F$,

$\therefore CA$ must fall on FD .

Thus the point **A** falls both on **ED** and **FD** ;

\therefore **A** falls on **D** which is the point in which **ED** and **FD** intersect.

Hence the two \triangle s coincide, and are \therefore equal in every respect.

Case II. Since two \angle s of the $\triangle ABC$ are respectively equal to two \angle s of the $\triangle DEF$,

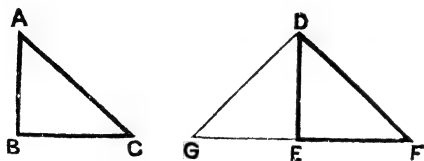
\therefore the remaining $\angle BAC$ of the one
= the remaining $\angle EDF$ of the other.
(*Th. 16, Cor. 2.*)

Thus we have the \angle s **A**, **B** respectively = the \angle s **D**, **E** and the side **AB** = the side **DE**. From this point we have to proceed as in Case I, by so applying the $\triangle ABC$ to the $\triangle DEF$ that **AB** may coincide with **DE**.

Case III. This is exactly similar to Case II, and we have to proceed by applying the $\triangle ABC$ to the $\triangle DEF$ so that **AC** may coincide with **DF**.

Theorem 19.

Two right-angled triangles which have their hypotenuses equal, and one side of the one equal to one side of the other, are equal in all respects



Let ABC and DEF be two right-angled Δ s, in which the hypotenuse $AC =$ the hypotenuse DF , and the side $AB =$ the side DE .

To prove that the two Δ s are equal in all respects.

Proof. Apply the ΔABC to the ΔDEF so that AB may coincide with DE to which it is equal, and C may fall on that side of DE which is remote from F .

Thus, let DEG be the new position of the ΔABC .

Now, since each of the \angle s DEF , DEG is a rt. \angle ,

$\therefore EG$ is in the same str. line with FE . (*Th. 2.*)

Hence DGF is a Δ , and since $DF = DG$. (*Hyp.*)

\therefore the $\angle DGF =$ the $\angle DFG$. (*Th. 5.*)

Hence, in the Δ s DEG , DEF , we have

(1) the $\angle DEG =$ the $\angle DEF$,

(2) the $\angle DGE =$ the $\angle DFE$. (*proved above.*)

and (3) DE common;

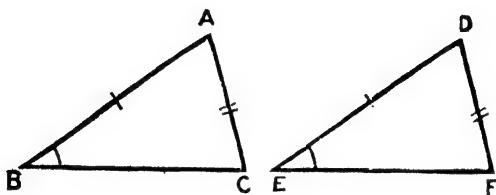
\therefore these two Δ s are equal in all respects. (*Th. 18.*)

That is, the Δ s ABC , DEF are equal in all respects.

Q. E. D.

Theorem 20.

If two triangles have two sides of the one respectively equal to two sides of the other, and have also the angles opposite to one pair of equal sides equal, then the angles opposite to the other pair of equal sides are either equal or supplementary.



Let the \triangle s ABC , DEF be such that $AB = DE$, $AC = DF$, and the $\angle ABC =$ the $\angle DEF$.

To prove that the \angle s ACB , DFE are either equal or supplementary.

Proof. The angle BAC must be *either* equal to the $\angle EDF$ or not.

Case I. When the $\angle BAC =$ the $\angle EDF$.

In the \triangle s ABC , DEF , we have

(i) $AB = DE$,

(ii) $AC = DF$,

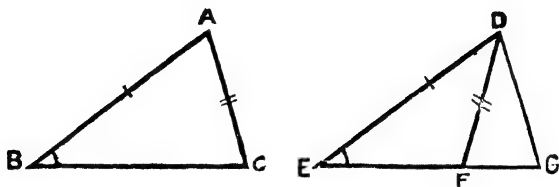
and (iii) the $\angle BAC =$ the $\angle EDF$;

\therefore the two \triangle s are congruent.

(Th. 4.)

Hence the $\angle ACB =$ the $\angle DFE$.

Case II. When the $\angle BAC$ is *not* equal to the $\angle EDF$.



On the same side of DE as F, suppose the $\angle EDG$ is made equal to the $\angle BAC$, and let EF, produced if necessary, meet DG in G.

Then in the \triangle s ABC, DEG, we have

(1) the $\angle ABC =$ the $\angle DEG$,

(2) the $\angle BAC =$ the $\angle EDG$, (Cons.)

and (3) $AB = DE$;

\therefore the two \triangle s are congruent. (Th. 18.)

Hence, $DG = AC = DF$;
and also the $\angle DGE =$ the $\angle ACB$. }

Hence the $\angle DFG =$ the $\angle DGE =$ the $\angle ACB$.

But the $\angle DFG$ is supplementary to the $\angle DFE$;

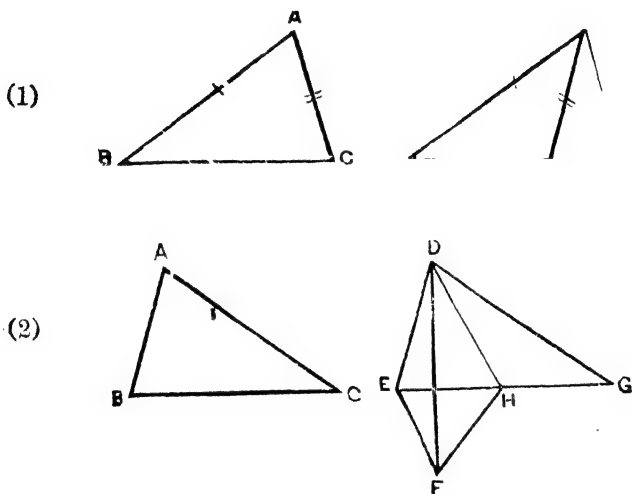
\therefore the $\angle ACB$ is supplementary to the $\angle DFE$.

Q. E. D.

Note. It is possible for the angle **ACB** and **DFE** to be supplementary *only when one of them is acute and the other obtuse.*

Theorem 21. (Euc. I. 24.)

If two triangles be such that two sides of the one are respectively equal to two sides of the other, but the angle contained by the two sides of the one is greater than the angle contained by the two sides of the other, then the base of that which has the greater angle is greater than the base of the other.



Let ABC , DEF be two \triangle s such that

$$AB = DE,$$

$$AC = DF,$$

but the $\angle BAC$ is greater than the $\angle EDF$.

To prove that the base BC is greater than the base EF .

Proof. Apply the $\triangle ABC$ to the $\triangle DEF$ so that A may fall on D and AB on DE .

Then since $\angle ADE = B \quad \therefore B$ coincides with E .

Let DG, EG be the new position of AC, BC .

(1) If EG passes through F as in fig. (1), then EG is $> EF$; i.e., BC is $> EF$.

(2) If EG does not pass through F , as in fig. (2), suppose the $\angle FDG$ is bisected by DH which meets EG at H .

Join FH .

Then in the \triangle s FDH, GDH , we have

(i) $FD = GD$,

(ii) DH common,

and (iii) the $\angle FDH =$ the $\angle GDH$; (Cons.)

\therefore the two triangles are congruent. (Th. 4.)

Hence, $FH = GH$,

and $\therefore EG = EH + HF$.

But the two sides EH, HF of the $\triangle EHF$ are together $> EF$;

$\therefore EG > EF$,

that is, $BC > EF$.

Q. E. D.

Cor. If two triangles ABC, DEF be such that AB is equal to DE , AC is equal to DF , but the base BC is greater than the base EF then the angle BAC is greater than the angle EDF .

The $\angle BAC$ cannot be less than the $\angle EDF$; for then BC would be less than EF , which is impossible.

Nor can the $\angle BAC$ be equal to the $\angle EDF$; for then BC would be equal to EF , which also is impossible.

\therefore the $\angle BAC$ must be $>$ the $\angle EDF$.

EXERCISE (7).

1. If the bisector of the vertical angle of a triangle is also perpendicular to the base, prove that the triangle is isosceles.

2. Prove that any point on the bisector of an angle is equidistant from the arms of the angle.

3. ABC is a triangle of which the side AB is greater than the side AC ; if D is the mid-point of BC and if AD is joined, prove that the angle ADB is an obtuse angle.

4. If a straight line AB is bisected at O , prove that the perpendiculars from A and B upon any other straight line through O are equal.

5. Prove that perpendiculars drawn from the extremities of the base of an isosceles triangle upon the opposite sides are equal.

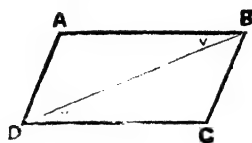
6. OB and OC are perpendiculars upon two straight lines AB and AC which meet at A ; if $OB = OC$, prove that AO is the bisector of the angle BAC .

7. ABC is a triangle of which the side AB is greater than the side AC ; if D is the mid-point of BC , and if any point E be taken on AD , prove that BE is greater than CE .

8. $ABCD$ is a quadrilateral such that the diagonal AC bisects each of the angles BAD , BCD . Prove that AC bisects BD at right angles.

Theorem 22. (Euc. I. 33.)

If a quadrilateral be such that two of its opposite sides are equal and parallel, then the other two sides also are equal and parallel.



Let ABCD be a quadrilateral such that two of its opposite sides AB, DC are equal and parallel.

To prove that the other two sides AD, BC are also equal and parallel.

Join BD.

Proof. Then, because AB and DC are parallel, and BD meets them,

\therefore the $\angle ABD =$ the alt. $\angle BDC$.

Now in the \triangle s ABD, CDB, we have

(1) $AB = CD$,

(2) BD common,

and (3) the $\angle ABD =$ the $\angle CDB$; (proved above.)

\therefore the two \triangle s are congruent. (Th. 4.)

Hence $AD = CB$; (i)

and the $\angle ADB =$ the $\angle CBD$.

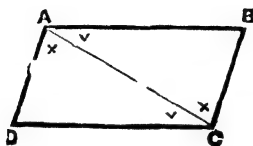
But these two \angle s are alt. \angle s;

\therefore AD and BC are parallel. (ii)

Thus, from (i) and (ii), AD and BC are both equal and parallel.

Theorem 23. (Euc. I. 34.)

The opposite sides and angles of a parallelogram are equal to one another, and each diagonal bisects the parallelogram



Let ABCD be a parallelogram, of which AC is a diagonal.

To prove that

- (i) $AB = DC$;
- (ii) $AD = BC$;
- (iii) the $\angle ABC =$ the $\angle ADC$;
- (iv) the $\angle BAD =$ the $\angle BCD$;
- (v) the $\triangle ABC =$ the $\triangle ADC$ in area.

Proof. Because AB, DC are parallel, and AC meets them,

\therefore the $\angle BAC =$ the alt. $\angle ACD$.

Also, because AD, BC are parallel, and AC meets them,

\therefore the $\angle DAC =$ the alt. $\angle ACB$.

Hence, in the \triangle s ABC, ADC, we have

- (1) the $\angle BAC =$ the $\angle ACD$,
 - (2) the $\angle BCA =$ the $\angle CAD$,
- } (proved above.)

and (3) AC common ;

\therefore the two \triangle s are congruent.

(Th. 18.)

Hence $\left. \begin{array}{l} AB = DC, \\ AD = BC, \\ \text{the } \angle ABC = \text{the } \angle ADC, \end{array} \right\}$

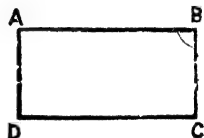
and the $\triangle ABC = \text{the } \triangle ADC$ in area.

Also, since the $\angle BAC = \text{the } \angle ACD$,
and the $\angle DAC = \text{the } \angle ACB$: } (*proved above*)

\therefore the whole $\angle BAD = \text{the whole } \angle BCD$. Q. E. D.

Cor. 1. *A rectangle has all its angles right angles.*

Let ABCD be a rectangle.
Then by definition, (i) it is a
parallelogram and (ii) it has one
angle, say the $\angle B$, a right angle.

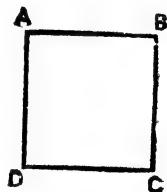


Since AB and DC are \parallel , the
 $\angle B + \text{the } \angle C = 2 \text{ rt. } \angle \text{ s}$; hence, the $\angle C$ is a rt. \angle .

Again, because ABCD is a par^m , the $\angle B = \text{the } \angle D$,
and the $\angle C = \text{the } \angle A$; hence the $\angle \text{ s } A$ and D also are
rt. $\angle \text{ s}$.

Cor. 2. *A square has all its sides equal and all its angles right angles.*

Let ABCD be a square; then by
definition, (i) it is a rectangle and (ii)
has two adjacent sides, say AB and BC,
equal.



Since it is a rectangle, \therefore all its
 $\angle \text{ s are rt. } \angle \text{ s}$.

Again, since the square is also a
 par^m ;

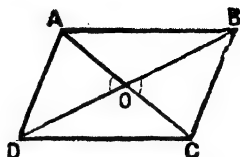
$\therefore AB = DC$ and $BC = AD$.

But $AB = BC$, by hypothesis;

\therefore the s **ha** also all its sides equal.

Cor. 3. *The diagonals of a parallelogram bisect one another.*

Let the diagonals AC, BD of the par^m. ABCD intersect at O.



To prove that $AO = OC$, and $BO = OD$.

In the \triangle s AOD, COB, we have

(1) the $\angle AOD =$ the $\angle COB$,

(2) the $\angle OAD =$ the alt. $\angle OCB$,

and (3) $AD = BC$;

(*opp. sides*)

\therefore the two \triangle s are congruent.

(*Th. 18.*)

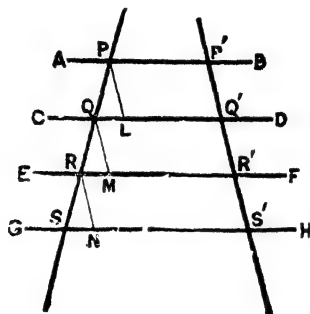
Hence $AO = OC$, and $BO = OD$.

Cor. 4. *A rhombus has all its sides equal.*

— — —

Theorem 24.

If there are three or more parallel straight lines, and the intercepts made by them on any straight line that cuts them are equal, then the corresponding intercepts on any other straight line that cuts them are also equal.



Let PS be a str. line which cuts the parallel str. lines AB, CD, EF, GH in P, Q, R, S so that $PQ = QR = RS$.

Let $P'S'$ be any other str. line which cuts the parallels in P', Q', R', S' .

To prove that $P'Q' = Q'R' = R'S'$.

Suppose PL, QM , and RN are drawn \parallel to $P'S'$, meeting CD in L, EF in M and GH in N .

PROOF. Since CD is \parallel to EF , and PR meets them,

\therefore the $\angle PQL =$ the corresponding $\angle QRM$.

Also, since PL and QM are parallel, each being \parallel to $P'R'$,

\therefore the $\angle QPL =$ the corresponding $\angle RQM$.

Then in the $\triangle s PQL$ and QRM , we have

(1) the $\angle PQL =$ the $\angle QRM$,

(2) the $\angle QPL =$ the $\angle RQM$,

and (3) $PQ = QR$;

Hence $PL = QM$, $\left\{ \right.$
 By a similar mode of reasoning, $QM = RN$. $\left. \right\}$

Now, the quadrilateral $PLQ'P'$ is a par^m , since its opp. sides are \parallel ,

$$\therefore PL = P'Q'. \quad (a) \quad (Th. 23.)$$

Similarly, the quadl. $QMR'Q'$ being a par^m ,

$$QM = Q'R'. \quad (b) \quad (Th. 23.)$$

And the quadl. $RNS'R'$ being a par^m ,

$$RN = R'S'. \quad (c) \quad (Th. 23.)$$

Hence, from (a), (b) and (c), $P'Q' = Q'R' = R'S'$.

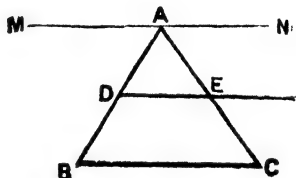
Q. E. D.

Cor. 1. *If D be the middle point of the side AB of a triangle ABC, then the straight line drawn through D parallel to BC will bisect AC.*

Let the str. line through D
 \parallel to BC intersect AC at E.

To prove that $AE = EC$.

If through A a str. line MN
 be supposed to be drawn \parallel to
 BC, then the parallels MN,
 DE, BC are cut by the two str. lines AB, AC.



Hence, the intercepts made on AB being equal, the corresponding intercepts on AC must be equal too.

Hence $AE = EC$.

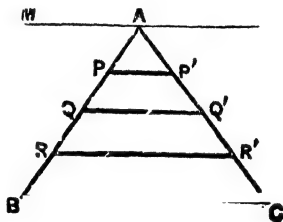
Cor. 2. *If D and E be the middle points of the sides AB and AC of a triangle ABC, then DE is parallel to BC.*

For, if DE be not \parallel to BC, let DE' be \parallel to it, meeting AC in E' . Then E' is the mid-point of AC, which is impossible. Hence DE is \parallel to BC.

Note. It is easy to see that $BC = 2DE$. For, if through E , EF be drawn parallel to AB to meet BC in F , then $DEFB$ is a par^m., and $\therefore BF = DE$. Also, since the Δ s ADE and EFC are congruent, $\therefore DE = FC$. Hence $BF = FC$, and $BC = 2BF = 2DE$.

Cor. 3. *If one side of a triangle be divided into any number equal parts, and if through the points of division, straight lines be drawn parallel to the base, then the points in which these parallels meet the other side will divide that side into the same number of equal parts.*

If the points P, Q, R divide the side AB of a triangle ABC into *four* equal parts; and if PP', QQ', RR' drawn \parallel to BC meet AC in P', Q', R' : then AC also is divided into *four* equal parts at these points.



Supposing MAN to be drawn \parallel to BC , we find that the five parallels make equal intercepts on the line AB which cuts them.

Hence the intercepts made by them on another line AC , which cuts them, must also be equal.

EXERCISE (8).

1. $ABCD$ is a parallelogram; if E and F are respectively the middle points of AB and CD , prove that the quadrilateral $AECF$ is a parallelogram.

2. Prove that a quadrilateral is a parallelogram,

(i) if its opposite sides are equal;

or (ii) if its opposite angles are equal;

or (iii) if its diagonals bisect one another.

3. If the diagonals of a parallelogram are equal, prove that it is a rectangle.

4. From any two points on a straight line perpendiculars are drawn to a parallel straight line ; prove that these perpendiculars are equal.

5. ABC and $A'B'C'$ are two triangles such that AB , AC are respectively equal and parallel to $A'B'$, $A'C'$. Prove that BC also is equal and parallel to $B'C'$.

6. Prove that the diagonals of a rhombus bisect each other at right angles.

7. If in a trapezium the non-parallel sides are equal, prove that the angles adjacent to either of the parallel sides are equal.

8. The straight line drawn through the middle point of a side of a triangle, parallel to the base, bisects the remaining side. Prove this without applying Theorem 21.

9. If D and E are respectively the middle points of the sides AB and AC of a triangle, prove that DE is parallel to BC , without employing the Indirect Method.

10. In example 9, prove that $DE = \frac{1}{2}BC$.

11. If the middle points of adjacent sides of any quadrilateral are joined, prove that the figure thus formed is a parallelogram.

12. If through the middle point of a side of a triangle a straight line be drawn parallel to the base, prove that it bisects any straight line drawn from the vertex to the base.

SECTION VI.

PROBLEMS.

In order that the constructions made may be *practically* as much accurate as possible, the student should provide himself with the following instruments :—

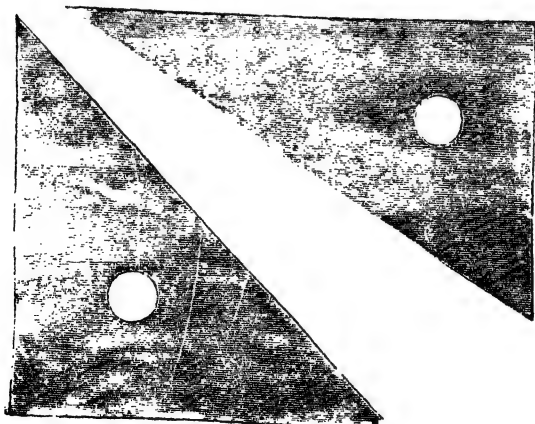
(1) **A Hard Pencil** ; well sharpened, so that the traces made may be as fine as possible.

(2) **A Pair of Compasses** (also called **Dividers**).

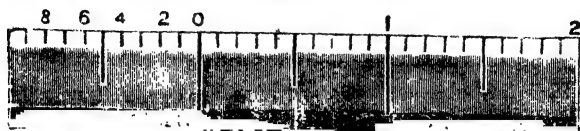


(3) **Pencil Compasses.**

(4) **Two Set-squares** ; one with angles of 45° and the other with angles of 60° and 30° .



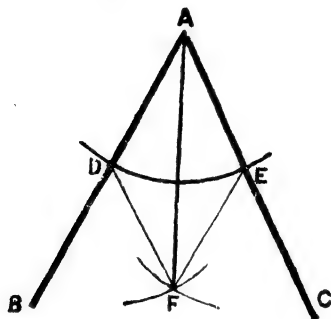
(5) A Graduated Flat Ruler ; shewing inches and tenths of an inch on one side, and centimetres and tenths of a centimetre on the other.



Note. In the following Problems, however, the necessary constructions can be effected without the aid of a *graduated* ruler ; simply a *str. ruler* will do. It is only in special cases, where the actual measurement of a line may be necessary, that a *graduated* ruler will be required.

Problem 1. (Euc. I. 9.)

To bisect a given angle.



Let BAC be the given angle.

It is required to divide it into two equal angles.

Cons. With centre A , and any radius, describe an arc of a circle cutting AB , AC at D and E respectively.

With centre D , and radius DE , describe an arc of a circle, on the side of DE remote from A ; also with centre E , and radius ED , describe another arc cutting the former at F .

Join AF .

Then AF is the bisector of the $\angle BAC$.

Proof. Join DF , EF .

In the \triangle s DAF , EAF , we have

(1) $AD = AE$, (radii of the same circle)

(2) $DF = EF$, (each being $= DE$)

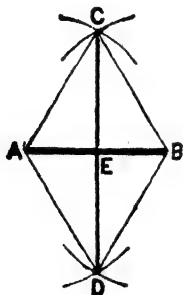
and (3) AF common;

\therefore the two \triangle s are equal in all respects.

Hence the $\angle DAF = \angle EAF$;
that is, the $\angle BAC$ is bisected by AF .

Q. E. F.

Note. The letter Q. E. F. at the end of a problem stand for *Quod Erat Faciendum*, that is, *which was to be done*.

Problem 2. (Euc. I. 10.)*To bisect a given finite straight line*

Let AB be the given finite str. line.

It is required to divide it into two equal parts.

Cons. With centre A , and radius AB , describe two arcs, one on each side of AB .

With centre B , and radius BA , describe two other arcs cutting the former at C and D .

Join CD , cutting AB at E .

Then E is the middle point of AB .

Proof. Join AC , BC , AD , BD .

In the \triangle s ACD , BCD , we have

$$(1) \quad AC = BC, \text{ (each being } = AB)$$

$$(2) \quad CD \text{ common,}$$

$$\text{and} \quad (3) \quad AD = BD : \text{ (each being } = AB)$$

\therefore the two \triangle s are equal in all respects.

Hence the $\angle ACD =$ the $\angle BCD$.

Again, in the \triangle s ACE , BCE , we have

$$(1) \quad AC = BC,$$

$$(2) \quad CE \text{ common,}$$

and $(3) \quad$ the $\angle ACE =$ the $\angle BCE$; (*proved above*);

\therefore the two \triangle s are equal in all respects.

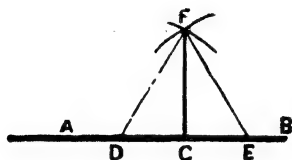
Hence $AE = EB$;

i.e., AB is bisected at E .

Q. E. F.

Problem 3. (Euc. I. 11.)

To draw a straight line perpendicular to a given straight line from a given point in it.



Let AB be the given str. line, and C the given point in it.

It is required to draw from C a str. line \perp to AB .

Cons. With centre C , and any convenient radius, describe a circle cutting AB at D and E .

With centres D and E , and radius DE , describe two arcs cutting each other at F .

Join CF .

Then CF is perpendicular to AB .

Proof. Join FD , FE .

In the \triangle s FDC , FEC , we have

$$(1) \quad FD = FE, \text{ (each being } = DE)$$

$$(2) \quad DC = EC, \quad (\text{Cons.})$$

and $(3) \quad FC$ common ;

\therefore the two \triangle s are equal in all respects.

Hence the $\angle FCD =$ the $\angle FCE$.

But these are adjacent angles ; and therefore each of them is a right angle.

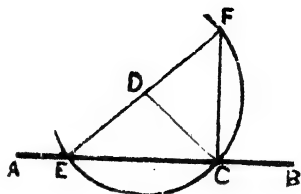
$\therefore CF$ is \perp to AB .

Q. E. F.

Alternative Method.

Cons. Take any point D outside AB .

With centre D , and radius DC , describe a circle cutting AB at E .



Join ED , and produce it to meet the circle at F . Join CF .

Then CF is \perp to AB .

Proof. Join DC .

Since $DE = DC$, \therefore the $\angle DCE =$ the $\angle DEC$.

Also $\therefore DF = DC$, \therefore the $\angle DCF =$ the $\angle DFC$.

\therefore the whole $\angle ECF =$ the $\angle FEC +$ the $\angle EFC$
 $=$ the $\angle BCF$.

And these are adjacent angles,

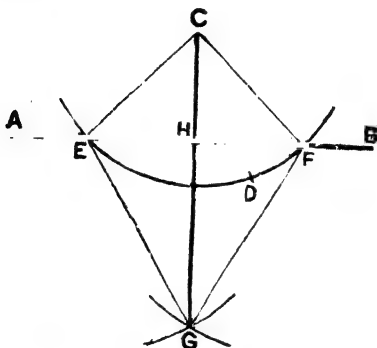
$\therefore CF$ is \perp to AB .

Note 1. That the angle ECF is a right angle may also be proved as follows : Since the $\angle ECF =$ the $\angle FEC +$ the $\angle EFC$;
 \therefore each side of this equality $=$ half the sum of the two sides $=$ half of two rt. \angle s $=$ one rt. angle.

Note 2. This method is applicable when the point C is near, or at one end of the straight line AB .

Problem 4. (EUC. I. 12.)

To draw a straight line perpendicular to a given straight line from a given point outside it.



Let AB be the given str. line, and C the given pt. outside it.

It is required to draw from C a straight line \perp to AB.

Cons. Take any point D on the side of AB remote from C.

With centre C, and radius CD, describe a circle cutting AB at E and F.

With centres E and F, and radius EF, describe two arcs, on the side of AB remote from C, cutting each other at G.

Join CG cutting AB at H.

Then CH is \perp to AB.

Proof. Join CE, CF, GE, GF.

In the \triangle s CEG, CFG, we have

$$(1) \quad CE = CF,$$

$$(2) \quad EG = FG, \text{ (each being } = EF)$$

and $(3) \quad CG$ common ;

\therefore the two \triangle s are equal in all respects.

Hence the $\angle ECG =$ the $\angle FCG$.

Again in the \triangle s CEH , CFH , we have

(1) $CE = CF$,

(2) CH is common,

and (3) the $\angle ECH =$ the $\angle FCH$:

\therefore the two \triangle s are equal in all respects.

Hence the $\angle CHE =$ the $\angle CHF$:

and these are adjacent angles. and \therefore each of them is a rt. angle.

$\therefore CH$ is \perp to AB .

Q. E. F.

Alternative Method.

Cons. Take any

point D in AB .

Join DC and bisect it at E .

With centre E , and radius EC , describe a circle cutting AB at D and F .

Join CF .

Then CF is \perp to AB .

Proof. Join EF .

Since $ED = EF$, \therefore the $\angle EFD =$ the $\angle EDF$.

Also since $EC = EF$, \therefore the $\angle EFC =$ the $\angle ECF$.

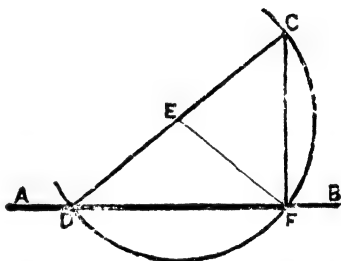
\therefore the whole $\angle DFC =$ the $\angle CDF +$ the $\angle DCF$
 $=$ the $\angle BFC$.

But these are adjacent \angle s, and \therefore each of them is a right angle.

Hence CF is \perp to AB .

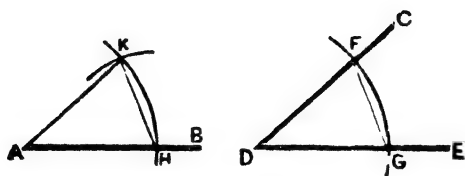
Q. E. F.

Note. This method is applicable when the point C is nearly opposite to one end of AB . The first method also may be applied by producing AB to any required length.



Problem 5. (Euc. I. 23.)

At a given point in a given straight line to make an angle equal to a given angle.



Let A be the given pt. in the given str. line AB , and CDE the given angle.

It is required to make at the pt. A in the str. line AB an \angle equal to the $\angle CDE$.

Cons. With centre D , and any radius, describe a circle cutting DC and DE at F and G respectively.

With centre A , and radius DG , describe an arc cutting AB at H ; with centre H , and radius GF , describe an arc cutting the former at K .

Join AK .

Then KAH will be the reqd. angle.

Proof. Join FG , KH .

In the Δ s KAH , FDG , we have

$$(1) \quad AH = DG, \quad (\text{Cons.})$$

$$(2) \quad AK = DF, \quad (\text{Cons.})$$

$$\text{and } (3) \quad HK = GF; \quad (\text{Cons.})$$

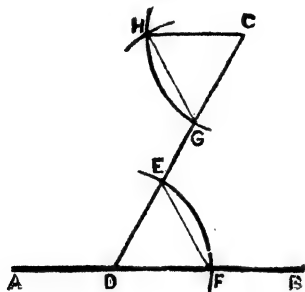
\therefore the two Δ s are equal in all respects.

Hence the $\angle KAH =$ the $\angle FDG$.

Q. E. F.

Problem 6. (Euc. I. 31.)

Through a given point to draw a straight line parallel to a given straight line.



Let C be the given point, and AB the given straight line.

It is required to draw through C a straight line parallel to AB .

Cons. Take any point D in AB , and join CD .

With centre D , and any radius, describe a circle cutting CD at E and DB at F .

With centre C , and radius DE , describe an arc, on the side of CD remote from B , cutting CD at G .

With centre G , and radius FE , describe an arc cutting the former at H .

Join CH .

Then CH is parallel to AB .

Proof. Join EF , HG .

In the \triangle s CHG , DFE , we have

$$(1) \quad CH = DF, \quad (\text{Cons.})$$

$$(2) \quad CG = DE, \quad (\text{Cons.})$$

$$\text{and } (3) \quad GH = FE: \quad (\text{Cons.})$$

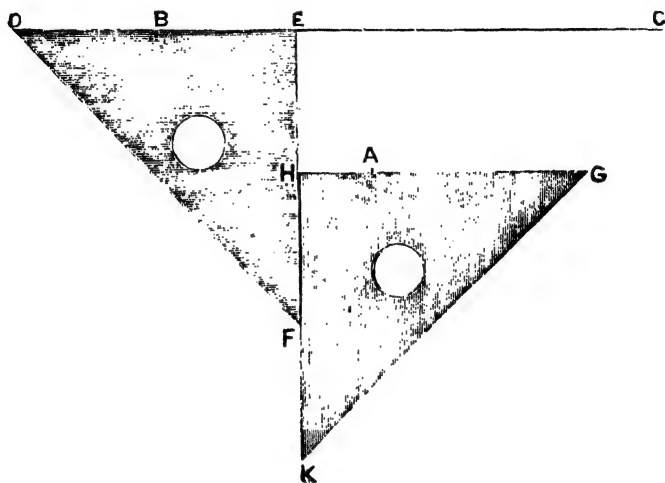
Hence the $\angle GCH =$ the $\angle EDF$, and these are alternate angles ;

$\therefore CH$ is \parallel to AB .

Q. E. F.

Alternative Method.

Let A be the given point, through which a str. line is to be drawn \parallel to the given str. line BC .



Cons. Place the Set-square DEF so that the edge DE may fall along BC .

Then slip the other Set-square GHK into the position shewn in the diagram, so that HG may pass through A .

Now trace a line along HG .

Then GH is \parallel to BC .

Proof. The $\angle DEF =$ the $\angle GHE$, each of them being a rt. \angle .

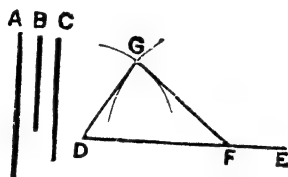
And these are alternate \angle s ;

$\therefore HG$ is \parallel to BC .

Q. E. F.

Problem 7. (EUC. I. 22.)

To construct a triangle having its sides equal to three given straight lines, any two of which are together greater than the third.



Let A, B, C be three given str. lines, of which A is the greatest, and any two of which are together greater than the third.

It is required to construct a triangle having its sides equal to the str. lines A, B, C.

Cons. Take any str. line DE, and from it cut off DF equal to A

With centre D, and radius B, describe a circle.

With centre F, and radius C, describe another circle cutting the former at G.

Join GD, GF.

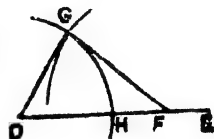
Then GDF is the reqd. \triangle ;

because, by construction, $\left. \begin{array}{l} DF = A, \\ DG = B, \\ FG = C. \end{array} \right\}$

and

Q. E. F.

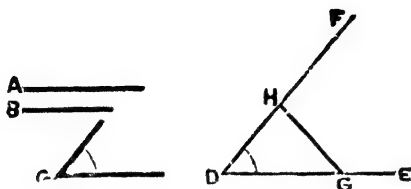
Note 1. If the circle described with centre D , and radius B , cuts DF at H , as in the following diagram, then $DH = B$. Now since B and C are together greater than DF , $\therefore C$ is greater than FH . Hence the circle described with centre F , and radius C , must evidently be partly inside and partly outside the former circle ; which shews that the second circle must cut the first. If B and C were together not greater than A , then the two circles would not cut each other, and the construction would fail.



Note 2. The circles would also cut at a point on the other side of DF . Hence, with the given parts, *two* triangles can be constructed one on either side of DF .

Problem 8.

To construct a triangle having given two sides and the included angle.



Let A, B be two given str. lines, and C a given angle.

It is required to construct a triangle having two of its sides equal to A and B, and having the angle included between these two sides equal to the $\angle C$.

Cons. Take any str. line DE.

At the point D in the str. line DE make the $\angle EDF$ = the $\angle C$. (Prob. 5.)

From DE cut off $DG = A$, and from DF cut off $DH = B$.

Join HG.

Then HDG is the \triangle required;

for $DH = B$,

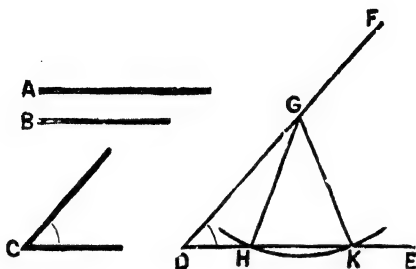
$DG = A$,

and the $\angle HDG = \text{the } \angle C$.

Q. E. F.

Problem 9.

To construct a triangle having given two sides and the angle opposite to one of them.



Let A, B be two given str. lines, and C a given angle.

It is required to construct a \triangle having two of its sides equal to A and B , and having the \angle opposite to the latter side = the $\angle C$.

Cons. Take any str. line DE .

At the pt. D in the str. line DE , make the $\angle EDF =$ the $\angle C$. (Prob. 5.)

From D cut off $DG = A$.

With centre G , and radius B , describe an arc of a \odot cutting DE at H and K .

If H and K are on the same side of D , as in the above diagram, join both GH and GK .

Then evidently *each* of the \triangle s GDH and GDK is the \triangle reqd. Q. E. F.

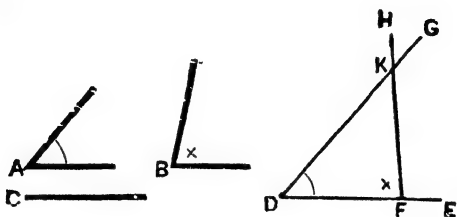
Note 1. There are two solutions to the problem, as above, only when B is less than A but greater than the perpendicular from G on DE . This is known as the **Ambiguous Case**.

Note 2. When B is less than the perpendicular from G on DE , the construction fails; or, in other words, the problem does not admit of a solution.

Note 3. If the given angle be a right angle, the problem amounts to constructing a right-angled triangle, of which the hypotenuse and one side are given.

Problem 10.

To construct a triangle having given two angles and the side adjacent to them.



Let A and B be two given \angle s, and C a given str. line.

It is reqd. to construct a Δ having two of its \angle s equal to the \angle s A and B, and the side adjacent to these angles = the str. line C.

Cons. Take any str. line DE. and from it cut off $DF = C$.

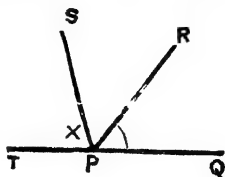
At the pt. D in the str. line DF, make the $\angle FDG =$ the $\angle A$. (Prob. 5.)

At the pt. F in the str. line FD, make the $\angle DFH =$ the $\angle B$. (Prob. 5.)

Let FH cut DG at K.

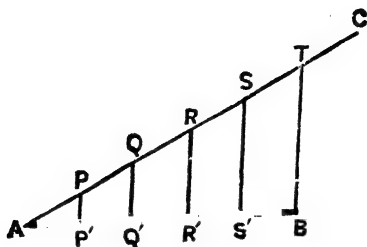
Then evidently, KDF is the Δ reqd. Q. E. F.

Note. If the $\angle DKF$ were given instead of the $\angle DFK$, the $\angle DFK$ might be found as follows:—Take any str. line PQ; at the pt. P in the str. line PQ, make the $\angle QPR =$ the $\angle A$; at the pt. P in the str. line PR, make the $\angle RPS =$ the $\angle DKF$; produce QP to any pt. T; then the $\angle SPT =$ the $\angle DFK$, for each of them is supplementary to the sum of the \angle s A and DKF.



Problem 11.

To divide a given straight line into any number of equal parts.



Let AB be the given str. line, and suppose that it is required to divide it into *five* equal parts.

Cons. From A draw a str. line AC , of unlimited length, making any angle with AB .

From AC cut off successively five *equal* parts of *any* length, AP , PQ , QR , RS , ST .

Join TB ; and through P , Q , R , S draw PP' , QQ' , RR' , SS' each \parallel to TB , meeting AB in P' , Q' , R' , S' respectively.

Then AB is divided into five equal parts at the points P' , Q' , R' , S' .

Proof. Since the side AT of the $\triangle ATB$ has been divided into five equal parts at the points P , Q , R , S , and through the points of division PP' , QQ' , RR' , SS' have been drawn \parallel to the base TB ;

\therefore the points P' , Q' , R' , S' also divide the other side AB into five equal parts. (Th. 24, Cor. 3.)

Q. E. F.

EXERCISE (9).

1. Take a straight line one inch in length, and construct an equilateral triangle on each side of it. Prove that the quadrilateral thus formed is a *rhombus*.

2. $\angle ABC$ is a right angle. Through B draw a straight line BD within the angle so that the angle ABD may be equal to one-third of the angle ABC.

3. Trisect a right angle.

4. Take a straight line one and a half inches long, and on it construct an isosceles triangle having each of the two sides $2\frac{1}{4}$ inches in length.

5. AB is a given straight line. Shew how to draw a straight line AC so that the angle BAC may be equal to 30° .

6. On a given straight line as hypotenuse, construct a right-angled triangle of which the acute angles are 30° and 60° .

7. Construct a right-angled triangle having given the hypotenuse and one side.

8. Construct an isosceles triangle having given the vertical angle and the length of the perpendicular from the vertex to the base.

9. From a given point outside a given straight line, draw a line making an angle with the given line equal to a given angle.

10. Construct a triangle of which the base angles shall be equal to two given angles, and the perpendicular from the vertex to the base equal to a given straight line.

11. Construct a triangle of which the two sides and the perpendicular from the vertex to the base are given.

12. Construct an equilateral triangle having given the perpendicular from the vertex to the base.

13. Construct a triangle having given the base, one of the angles at the base, and the sum of the sides.

14. Two sides of a triangle are respectively 3 inches and 2 inches in length, and the angle opposite to the latter side is equal to 30° ; construct the triangle.

Also construct the triangle when the length of the latter side is $3\frac{1}{2}$ inches.

Shew that in the former case there are two solutions, whilst in the latter there is only one.

15. If the side BA of a triangle ABC be produced to D so that $AD=AC$, and if DC be joined, prove that the angle BDC is half of the angle BAC. Hence shew how to construct a triangle having given the base, the vertical angle, and the sum of the sides.

16. ABC is a triangle. If D be taken on CB produced such that $BD=AB$, and if E be taken on BC produced such that $CE=AC$, and if AD and AE be joined; prove that the angle ADE is half of the angle ABC, and the angle AED is half of the angle ABC. Hence shew how to construct a triangle having given the *perimeter* and the base angles. (The sum of the three sides of a triangle is called its *perimeter*.)

17. Take a straight line 4 inches long, and divide it into seven equal parts.

SECTION VII.

LOCI.

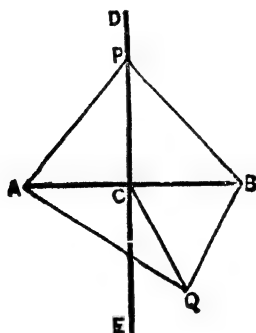
When a point moves and its motion is restricted by some given condition, the path traced out by the point is called its **locus**.

For instance, when a point moves so that its distance from a fixed point is always the same, its locus is evidently the circumference of a circle of which the fixed point is the centre and the given distance, the **radius**.

Note. When any number of points, each of which satisfies a given condition, lie on a particular line, that line is also called the locus of those *points*. From this point of view, the locus of points, each of which lies at a given distance from a given point, is the circumference of the circle which is described with the given point as centre and the given distance as radius.

Theorem 25.

The locus of a point which is equidistant from two fixed points is the perpendicular bisector of the straight line joining the two fixed points.



Let **A** and **B** be two fixed points, let **C** be the mid-point of the str. line **AB**.

Let DE be the unlimited str. line which passes through C and is perpendicular to AB .

To prove that if a point moves so that its distances from the points A and B are always equal, the only path along which it can move is the str. line DE .

Proof. (i) Take any pt. P on the str. line DE .

Join PA , PB .

In the \triangle s PCA , PCB , we have

$$(1) \quad AC = CB,$$

$$(2) \quad PC \text{ common,}$$

$$\text{and} \quad (3) \quad \angle PCA = \angle PCB;$$

\therefore the two \triangle s are equal in all respects.

Hence $PA = PB$.

Thus, *every point* on the str. line DE is equidistant from A and B (a)

(ii) Again, it may be proved that no pt. outside the str. line DE is equidistant from A and B .

For, if possible, let the pt. Q , which is *not* on DE , be equidistant from A and B .

Join QA , QB , QC .

Then in the \triangle s QCA , QCB , we have

$$(1) \quad QA = QB,$$

$$(2) \quad CA = CB,$$

$$\text{and} \quad (3) \quad QC \text{ common};$$

\therefore these two \triangle s are equal in all respects.

Hence the $\angle QCA = \angle QCB$;

and $\therefore QC$ is \perp to AB , which is impossible.

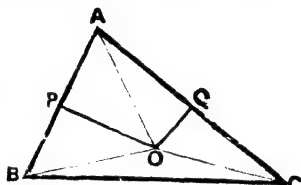
Thus *no point outside DE can be equidistant from A and B* (β)

From (α) and (β) we see that *every pt. on the str. line DE, and no other pt., is equidistant from A and B.*

Hence, the only path that the moving point can describe is the unlimited str. line DE. Q. E. D.

Cor. *The point of intersection of the perpendicular bisectors of the straight lines AB and AC is equidistant from the points A, B, C.*

Let PO and QO, which meet at O, be the \perp bisectors of AB and AC respectively.



Now, O being a pt. on PO, $OA = OB$. }

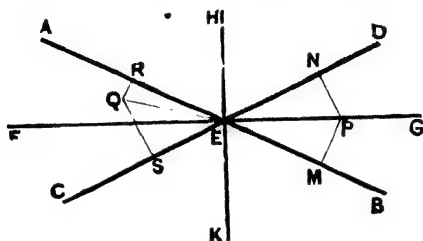
Also, O being a pt. on QO, $OA = OC$. }

Hence OA, OB, OC are equal to one another.

Note The perpendicular bisector of the straight line AB may also be said to be the locus of *points* which are equidistant from A and B.

Theorem 26.

The locus of a point which is equidistant from two intersecting straight lines consists of the pair of straight lines which bisect the angles between the two given lines.



Let AB, CD be two given str. lines which intersect at E .

Let FG be the bisector of the \angle s AEC and BED ; and let HK be the bisector of the \angle s AED and BEC .

To prove that a point which moves so that its distances from the str. lines AB and CD are always equal, must move along one or other of the two lines FG and HK .

Proof. (i) Take *any* pt. P on the line FG .

From P draw FM and $PN \perp$ to AB and CD respectively.

In the \triangle s PEM, PEN , we have

(1) the $\angle PEM =$ the $\angle PEN$,

(2) the $\angle PME =$ the $\angle PNE$, (*Rt. \angle s.*)

and (3) the side EP common;

\therefore the two \triangle s are equal in all respects.

Hence $PM = PN$.

Thus, *every* pt. on the line FG is equidistant from the lines AB and CD ;

and similarly *every pt.* on the line **HK** is equidistant from **AB** and **CD** ; (a)

(ii) Again, it may be proved that no pt. which lies outside **FG** and **HK**, is equidistant from **AB** and **CD**.

For, if possible, let the pt. **Q**, which is *not* on **FG** or **HK**, be equidistant from **AB** and **CD**.

From **Q** draw **QR**, **QS** \perp to **AB** and **CD** ; and join **QE**. Then, by hypothesis, **QR** = **QS**.

Now, in the *right-angled* Δ s **QRE**, **QSE**, we have

(1) the hypotenuse **QE** common,

and (2) **QR** = **QS** ;

\therefore the two Δ s are equal in all respects. (*Th. 19.*)

Hence, the \angle **QER** = the \angle **QES** ;

i.e., **QE** is the bisector of the \angle **AEC**, which is impossible.

Thus, *no* pt. which lies outside the lines **FG** and **HK**, can be equidistant from **AB** and **CD** (β)

From (α) and (β) we see that *every* pt. on the lines **FG** and **HK**, and *no other pt.*, is equidistant from **AB** and **CD**.

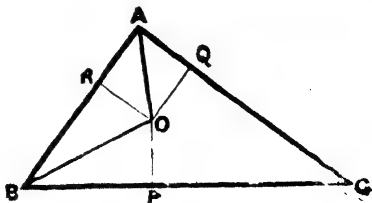
Hence the lines **FG** and **HK** form the locus of a point which moves under the proposed condition.

Q. E. D.

Cor The point of intersection of the bisectors of the angles **A** and **B** of a triangle **ABC** is equidistant from the sides of the triangle.

Let **AO** and **BO**, which meet at **O**, be the bisectors of the \angle s **A** and **B**.

From **O** draw **OP**, **OQ**, **OR** \perp s to **BC**, **CA**, **AB** respectively.



Since O is on the bisector of the $\angle A$, $\therefore OR = OQ$.
 Also, since O is on the bisector of the $\angle B$, $\therefore OR = OP$.

Hence, OP, OQ, OR are equal to one another;
i.e., the pt. O is equidistant from the sides BC, CA, AB .

Note. The lines FG and HK may also be said to be the locus of points which are equidistant from AB and CD .

EXERCISE (10).

1. Prove that the locus of a point which moves in such a manner that its distance from a given straight line is always the same, consists of two straight lines parallel to the given line, one on either side of it.

2. What is the locus of points which are equidistant from two given points? Find a point in a given straight line so that its distances from two given points outside the line may be equal.

3. What is the locus of points which are equally distant from two intersecting straight lines? Find a point in a given straight line so that it may be equally distant from two intersecting straight lines.

4. Find a point which is equally distant from two fixed points and is also at a given distance from a third fixed point. When is this problem impossible?

5. Find a point which is at a distance of 2 inches from one fixed point and 3 inches from another. When is this problem impossible?

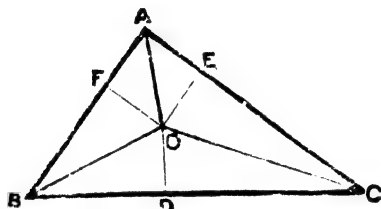
6. AB and AC are two fixed and unlimited straight lines. Find a point which is at a distance of 2 inches from AB and 3 inches from AC .

7. Find a point in a given straight line at a given distance from another given straight line.

SECTION VIII.

MISCELLANEOUS PROPOSITIONS.

1. *The bisectors of the angles of a triangle meet at a point.*



Let ABC be a \triangle . and let the bisectors of the \angle s B and C meet at O .

Join AO .

To prove that AO is the bisector of the $\angle A$.

From O draw $OD, OE, OF \perp$ to BC, CA, AB respectively.

Proof. Because BO bisects the $\angle ABC$,

\therefore any pt. in BO is equidistant from BC and BA .

(Th. 26.)

$\therefore OD = OF$.

For a similar reason, any pt. in CO is equidistant from CB and CA ;

$\therefore OD = OE$.

Hence

$OF = OE$.

$\therefore O$ is on the bisector of the $\angle BAC$; (Th. 26.)

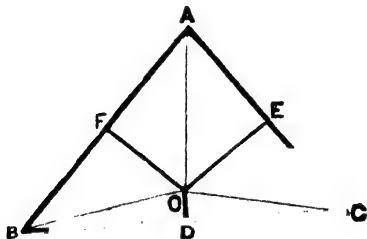
i.e., AO is on the bisector of the $\angle BAC$.

Thus, the bisectors of the \angle s A, B, C meet at a pt.

Q. E. D.

Note. When three or more straight lines meet at a point, they are said to be **Concurrent**. So the above proposition might be stated as follows :—*The bisectors of the angles of a triangle are concurrent.*

2. *The perpendiculars drawn to the sides of a triangle from their middle points are concurrent.*



Let ABC be a \triangle , and D, E, F be the mid-points of BC, CA, AB respectively.

From F and E draw \perp s to AB, AC meeting at O.

Join OD.

To prove that OD is \perp to BC.

Join OA, OB, OC.

Proof. Because O is on the perpendicular bisector of AB,

$$\therefore OA = OB. \quad (Th. 25.)$$

For a similar reason,

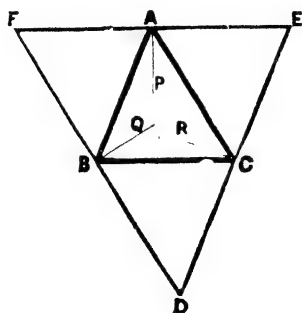
$$OA = OC.$$

Hence $OB = OC$;

and \therefore O is on the \perp bisector of BC. (Th. 25.)

Hence OD is \perp to BC. Q. E. D.

3. *The perpendiculars drawn from the vertices of a triangle to the opposite sides are concurrent.*



Let ABC be a \triangle , and let AP , BQ , CR , when produced, be \perp to BC , CA , AB respectively.

To prove that AP , BQ , CR (when produced) are concurrent.

Through A , B and C , draw str. lines \parallel to BC , CA and AB respectively, thus forming the $\triangle DEF$.

Proof. Since FE is \parallel to BC , and FD is \parallel to AC ,

$\therefore AFBC$ is a par^m .

$\therefore FA = BC$.

Similarly, $AE = BC$.

Hence, $FA = AE$; and $\therefore A$ is the mid-point of EF .

Similarly, B is the mid-pt. of FD , and C is the mid-pt. of DE .

Now, since BC is \parallel to FE , and AP produced is \perp to BC ,

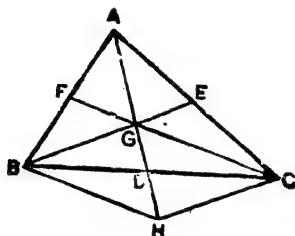
$\therefore AP$ is also \perp to FE .

Similarly, BQ is \perp to FD and CR is \perp to DE .

Thus, AP , BQ and CR are \perp s to the sides of the $\triangle EFD$ at their middle points; and \therefore they are concurrent.
(*Last Prop.*)

Q. E. D.

4. *The medians of a triangle are concurrent; the common point dividing each median into two parts, of which the part nearer the vertex is double the other.*



Let E, F be the mid-pts. of the sides CA and AB of the $\triangle ABC$, and let the medians BE, CF intersect at G. Join AG, and produce it to meet BC at D.

To prove that D is the mid-pt. of BC.

Through B draw BH \parallel to FC, and produce AD to meet BH at H.

Join CH.

Proof. (1). In the $\triangle ABH$, F is the mid-pt. of AB, and FG is \parallel to BH ;

\therefore G is the mid-pt. of AH. (*Th. 24, Cor. 1.*)

Again, in the $\triangle AHC$, G and E are the mid-pts. of the sides AE and AC ;

\therefore GE is \parallel to HC. (*Th. 24, Cor. 2.*)

Thus the opp. sides of the quadl. BGCH are \parallel , and \therefore is a par^m.

But the diagonals of a par^m. bisect one another ;

\therefore D is the mid-pt. of BC.

Thus the three medians AD, BE, CF meet at the point G. Q. E. D.

(2) Since D is the mid-pt. of GH,

\therefore GH = 2GD.

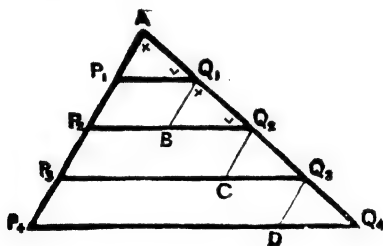
But $AG = GH$, $\therefore AG = 2GD$.

Also, $BG = HC = 2GE$; }
and, $CG = HB = 2GF$; } (Th. 24, Cor. 2, Note.)

Q. E. D.

Note. The point of intersection of the medians of a triangle is called the **Centroid** of the triangle. Thus, in the above figure, the point **G** is the *Centroid* of the $\triangle ABC$.

5. AP_1Q_1 is a triangle; from AP_1 produced, P_1P_2 , P_2P_3 , P_3P_4 are cut off each equal to AP_1 ; P_2Q_2 , P_3Q_3 , P_4Q_4 are drawn \parallel to P_1Q_1 , meeting AQ_1 produced in Q_2 , Q_3 , Q_4 . Prove that $P_2Q_2 = 2P_1Q_1$, $P_3Q_3 = 3P_1Q_1$, $P_4Q_4 = 4P_1Q_1$.



Draw Q_1B , Q_2C , Q_3D each \parallel to AP_4 , meeting P_2Q_2 , P_3Q_3 , P_4Q_4 in B , C , D respectively.

Proof. AQ_1 , Q_1Q_2 , Q_2Q_3 , Q_3Q_4 are equal to one another. (Th. 24, Cor. 3.)

Now, since the \triangle s AP_1Q_1 , Q_1BQ_2 are congruent,

$$\therefore P_1Q_1 = BQ_2 \quad \dots \quad (\alpha)$$

Similarly, $BQ_2 = CQ_3$, and $CQ_3 = DQ_4$.

Thus, P_1Q_1 , BQ_2 , CQ_3 , DQ_4 are equal to one another.

Again, since the quadl. $P_1P_2BQ_1$ is a par^m.,

$$\therefore P_1Q_1 = P_2B \quad \dots \quad (\beta)$$

Similarly, $P_2Q_2 = P_3C$ and $P_3Q_3 = P_4D$.

(1) From (a) and (β), $P_2Q_2 = 2P_1Q_1$.

(2) $P_3Q_3 = P_3C + CQ_3 = P_2Q_2 + P_1Q_1 = 3P_1Q_1$.

(3) $P_4Q_4 = P_4D + DQ_4 = P_3Q_3 + P_1Q_1 = 4P_1Q_1$.

Q. E. D.

Note. Hence, if DE be drawn parallel to the base BC of a triangle ABC , meeting AB , AC in D and E , the following conclusions are obvious :—

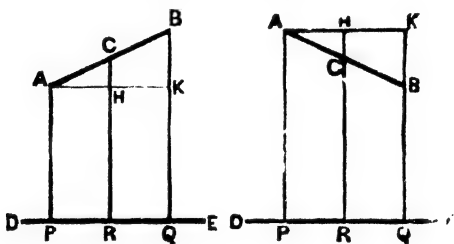
(1) if $AB = 2AD$, then $BC = 2DE$;

(2) if $AB = 3AD$, then $BC = 3DE$;

(3) if $AB = 4AD$, then $BC = 4DE$; and so on.

6. C is a point in a finite straight line AB ; DE is an unlimited straight line such that the points A and B are on one side of it; AP , BQ , CR are perpendiculars to DE . Prove that (1) if C is the middle point of AB , then $2CR = AP + BQ$; (2) if $BC = 2AC$, then $3CR = 2AP + BQ$.

(1) Let C be the middle point of AB .



Through A draw $AK \parallel$ to DE , meeting RC , QB , or those lines produced, in H and K .

The quadls. $APRH$ and $HRQK$ are par^m.;

$\therefore AP = HR = KQ$.

Again, since $AB = 2AC$, and $CH \parallel$ to BK ,

$BK = 2CH$.

(Last Prop., Note.)

Hence, in fig. (1),

$$2CR = 2HR + 2CH$$

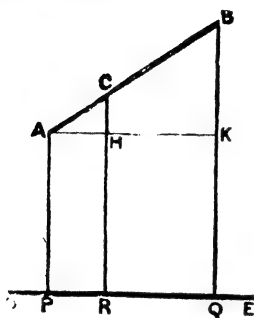
$$= AP + KQ + BK = AP + BQ.$$

and in fig. (2),

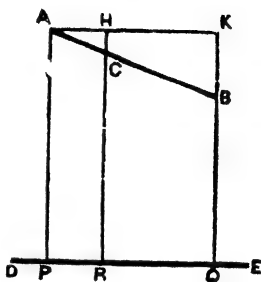
$$2CR = 2HR - 2CH$$

$$= AP + KQ - BK = AP + BQ.$$

(2) Let BC be $= 2AC$.



(1)



(2)

Through A draw $AK \parallel$ to DE , meeting RC , QB , or those lines produced, in H and K .

The quadrs. $APRH$ and $HRQK$ are par^m;

$$\therefore AP = HR = KQ.$$

Again, since $AB = 3AC$, and CH is \parallel to BK ,

$$\therefore BK = 3CH. \quad (\text{Last Prop., Note.})$$

Hence in fig. (1),

$$3CR = 3HR + 3CH$$

$$= 2AP + KQ + BK = 2AP + BQ.$$

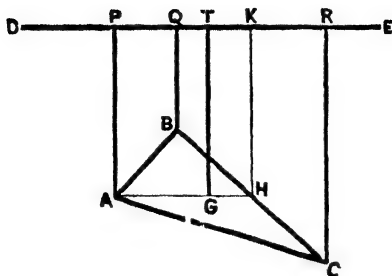
And in fig. (2),

$$3CR = 3HR - 3CH$$

$$= 2AP + KQ - BK = 2AP + BQ.$$

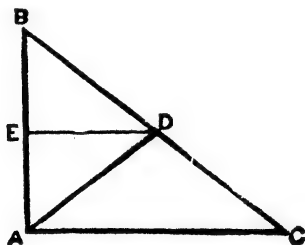
Q. E. D.

Note. Hence, if G be the *centroid* of a triangle ABC , and if AP , BQ , CR , GT be perpendiculars to an unlimited straight line, DE , which is such that the points A , B , C are all on one side of it then $3GT = AP + BQ + CR$.



For, let H be the mid-pt. of BC , and draw $HK \perp$ to DE ; then G is on AH , so that $AG = 2GH$. Hence $3GT = AP + 2HK = AP + BQ + CR$.

7. If D be the middle point of the hypotenuse BC of a right-angled triangle ABC , prove that $DA = \frac{1}{2}BC$.



Through D draw $DE \perp$ to AB ; then, since DE and CA are both \perp to AB , $\therefore DE$ is \parallel to CA .

Hence, since D is the mid-pt. of BC , E also is the mid-pt. of BA ; (Th. 24, Cor. 1.)

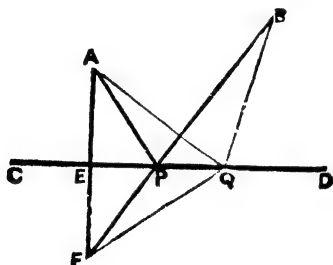
$\therefore ED$ is the \perp bisector of BA .

Hence, $DB = DA$.

But $DB = \frac{1}{2}BC$; $\therefore DA = \frac{1}{2}BC$.

Q. E. D.

8. *A and B are two points on the same side of an unlimited straight line CD. P is a point in CD such that the angles APC and BPD are equal. Prove that, of all points on the straight line CD, P is that the sum of whose distances from A and B is the least.*



Let Q be any other pt. in CD.

Join AQ, BQ.

To prove that $AQ + QB > AP + PB$.

Produce BP to F, making $PF = PA$; join AF, cutting CD in E.

Join FQ.

Proof. In the \triangle s APE, FPE, we have

(1) $AP = PF$,

(2) PE common,

and (3) the $\angle APE = \text{the } \angle EPF$

(\therefore each of them = the $\angle BPD$),

\therefore the two \triangle s are congruent.

Hence $AE = EF$, and EP is \perp to AF.

Thus, CD is the \perp bisector of the str. line AF.

Hence, $QA = QF$;

and $\therefore AQ + QB = FQ + QB$.

Also $AP + PB = FB$.

(Cons.)

But $FQ + QB$ is $> FB$,

$\therefore AQ + QB$ is $> AP + PB$.

Q. E. D.

Note. It is easy to see that there is only *one* point in **CD** such that the str. lines drawn from it to **A** and **B** are *equally* inclined to **CD**.

EXERCISE (11).

1. **ABC** is a triangle of which the side **AC** is greater than the side **AB**. **AD** is drawn bisecting the angle **BAC**, and meeting **BC** in **D**. Prove that **CD** is greater than **BD**.

2. If **D** be the middle point of the base **BC** of a triangle **ABC**, and if **AD** be joined, prove that $AB + AC > 2AD$.

3. Through a given point draw a straight line such that the perpendiculars on it from two fixed points may be on opposite side of it and equal to each other.

4. **ABCD** is a parallelogram ; **E** and **F** are the mid-points of **AB** and **CD** respectively. If **DE**, **BF** intersect the diagonal **AC** at **G** and **H**, prove that $AG = GH = HC$.

5. Prove that the medians of a triangle meet at a point which is a point of trisection on each of them. Hence find a method of trisecting a given straight line.

6. Construct a triangle having given the base, the perpendicular from the vertex to the base, and the line joining the vertex to the mid-point of the base.

7. **ABC** is a triangle. If **AB**, **AC** be produced to **D** and **E** respectively, prove that the bisectors of the angles **DBC**, **ECB** meet at a point which lies on the bisector of the angle **BAC**.

8. Through a given point draw a straight line making with a fixed straight line an angle equal to a given angle. Hence show how to construct a triangle of which the base angles and the perpendicular from the vertex to the base are given.

9. Construct a right-angled triangle having given the hypotenuse and the sum of the sides.

10. Prove that the locus of the vertices of all right-angled triangles which have a common hypotenuse is a circle.

11. Construct an isosceles triangle having given the base, and the sum of one of the equal sides and the perpendicular from the vertex to the base.

12. AB, AC are two given straight lines ; through a given pt. D between them, draw a straight line meeting AB, AC in G and H , so that GH may be bisected at D .

13. OA, OB, OC are three given straight lines, of which OB lies between OA and OC . Draw a straight line terminated by OA and OC , and bisected by OB .

14. $ABCD$ is a parallelogram, and MN is an unlimited straight line outside it. If AP, BQ, CR, DS be perpendiculars to MN , prove that $AP + CR = BQ + DS$.

15. If C is the middle point of a straight line AB , and D is any point in CB , prove that $AD - DB = 2CD$.

Hence prove that if C is the mid-point of a given straight line AB , where the points A and B are on opposite sides of an unlimited straight line DE , and if AP, BQ, CR be perpendiculars to DE , and $AP > BQ$, then $AP - BQ = 2CR$.

16. Construct a triangle, having given the base, an angle at the base, and the difference of the sides.

17. BC is the base of an isosceles triangle ABC , EAD is drawn parallel to BC , and any point P is taken on AD . Prove that $BP + PC > BA + AC$.

Prove also that if Q be *any* point above the line DE ,
 $BQ + QC > BA + AC$.

Hence prove that if R be *any* point *above* BC , such that
 $BR + RC = BA + AC$, then R must be *below* DE .

18. Construct a triangle, having given the middle points of the sides.

19. Of all triangles which stand on the same base and which have their vertices in a straight line parallel to the base, prove that the triangle which is isosceles has the *least* perimeter.

20. Prove that any side of a triangle is greater than the difference of the other two sides. Hence prove that if P is a point in an unlimited straight line MN , such that PA and PB are equally inclined to it, where A and B are two fixed points on opposite sides of MN , then of all points in the given straight line, P is that the *difference* of whose distances from A and B is the *greatest*.

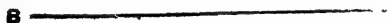
BOOK II

Areas.

SECTION I.

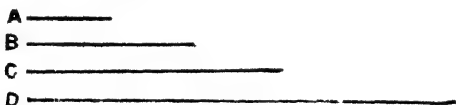
FUNDAMENTAL IDEAS AND DEFINITIONS.

1. If two straight lines A and B be such that B can be divided into a number of parts each of which is equal to A then the line A is said to be a **measure** of the line B.



Note 1. When B is divisible into parts each of which is equal to A it may be said that B *contains A an exact number of times*. Hence, we may say that *one straight line is a measure of another when the latter contains the former an exact number of times*.

2. If two or more straight lines, B, C, D be such that each of them contains a given straight line A an exact number of times, then A is said to be a **common measure** of B, C, D.



Note 2. In the above diagram A is contained twice in B, three times in C, and five times in D; i.e., an *exact number* of times in each. Hence A is a *common measure* of B, C, D.

3. Sizes, or lengths of different straight lines are ascertained by finding how often each of them contains any smaller straight line arbitrarily chosen.

Note 3. To *measure a straight line* is to ascertain its length in the manner above indicated.

4. Any convenient straight line that is arbitrarily chosen for the purpose of measuring other straight lines, is called the **unit of length**.

Note 1. The *number* of times that any given straight line contains the unit of length is called the **numerical measure** of the straight line, and the given line is said to be *represented* by that number.

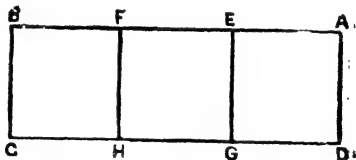
Note 2. When we say that one inch is the unit of length, we mean that *a straight line whose length is one inch* has been taken as the unit.

5. The amount of surface enclosed by the bounding lines of a figure is called its **area**.

6. The **unit of area** is the area of a square of which a side is equal to the unit of length.

Note 1. Let $ABCD$ be a rectangle of which the side $BC =$ the unit of length, and the side $AB =$ three times the unit. Then the area of the rectangle is equal

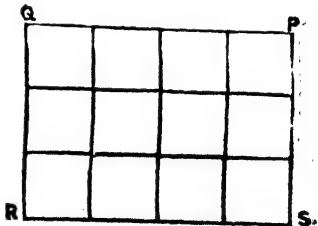
to *three* units of area. For, if AB be divided into three equal parts at E and F , and if through the points of division, EG and FH be drawn \parallel to BC , then evidently each of the figures BH , FG and ED



is a square of which a side = the unit of length.

Note 2. If $PQRS$ be a rectangle of which the side $PQ = 4$ units of length, and the side $QR = 3$ units of length, then the area of the rectangle is equal to 12 (i.e. 3×4) units of area.

For, if PQ be divided into 4 equal parts, and QR into 3 equal parts, and if through the pts. of division in either side, lines be drawn parallel to the other; then the whole rectangle is divided



into 3 rows of squares, a side of each square being equal to the unit of length. There are 4 squares in each row; the total number of squares = $3 \times 4 = 12$, and hence the area of the rectangle = 12 units of area.

Note 3. Hence, generally speaking, if the adjacent sides of a rectangle be equal to a and b units of length respectively (a and b being integers), then the area of the rectangle = ab units of area; and if a side of a square be equal to a units of length, the area of the square = a^2 units of area.

Note 4. Of two adjacent sides of a rectangle, the length of the greater side is called the *length of the rectangle*, and the length of the smaller side is called its *breadth*.

7. A Square inch is the *area* of a square of which a side is one inch in length.

Note 1. There is a clear distinction between "two square inches" and "two inches square." The former means an area twice as large as the area of a square of which a side is one inch, whilst the latter means a *square* of which a side is two inches in length. In fact, the area of a "two inches square" is *four square inches*, as can be easily seen by a diagram.

Note 2. If $ABCD$ be a rectangle of which the sides AB , BC are respectively $4\frac{2}{3}$ inches and $3\frac{1}{5}$ inches in length, then its area is equal to $(4\frac{2}{3} \times 3\frac{1}{5})$ square inches.

For, taking $\frac{1}{15}$ of an inch as the unit of length, AB and BC are found to contain 70 and 48 units respectively; and \therefore the required area = 70×48 units of area. But as 15 units of length = one inch, one sq. inch = 15×15 units of area. Hence, the number of sq. inches in the area of the rectangle = $\frac{70 \times 48}{15 \times 15} = \frac{1}{3} \times \frac{16}{5} = 4\frac{2}{3} \times 3\frac{1}{5}$, which proves the proposition.

8. The perpendicular drawn from the vertex of a triangle to the base, or the base produced, is called the *altitude*, or *height*, of the triangle.

Note 1. Triangles are said to be *between the same parallels* when their bases lie in one straight line and the vertices in another which is parallel to the former.

Note 2. It may be easily shewn that if two triangles are on the same base and between the same parallels, their altitudes are equal.

Note 3. Triangles having *equal* altitudes are said to have the *same altitude*.

9. The perpendicular drawn to the base of a parallelogram from *any* point in the opposite side is called the *altitude*, or *height*, of the parallelogram.

Note 1. Parallelograms are said to be *between the same parallels* when their bases lie in one straight line, and the sides opposite to the bases in another which is parallel to the former.

Note 2. It may be easily shewn that if two parallelograms are on the same base and between the same parallels their altitudes are equal

Note 3. Parallelograms having *equal altitudes* are said to have the *same altitude*.

10. A rectangle of which two adjacent sides are AB and BC , is said to be **the rectangle contained by AB and BC** .

Note 1. A rectangle of which two adjacent sides are equal to two given straight lines P and Q , is also said to be **the rectangle contained by P and Q** .

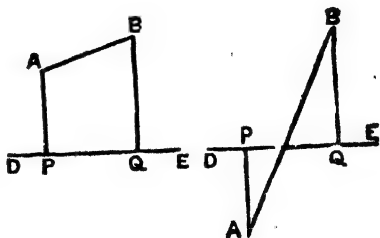
Note 2. "The rectangle contained by AB and BC " is briefly written as "the rect. AB, BC ;" or more briefly, as " AB, BC ".

11. A straight line AB is said to be divided **internally** or **externally**, at any point D , according as the point D is taken on AB , or on AB produced. In either case, AD and DB are called the *segments* of the line AB .

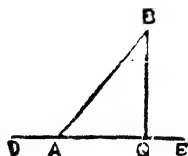


Note. In the case of **external division**, AD and DB are regarded as the segments of the line AB only in the sense that they are the distances of D from A and B respectively. In the case of **internal division**, AD and DB are not only the distances of D from A and B but they also together make up AB .

12. If from the extremities of a straight line AB , perpendiculars AP and BQ be drawn to an unlimited straight line DE , then PQ is said to be the **Projection** of AB on DE .



Note. If A is on the straight line DE , then AQ becomes the projection.



EXERCISE (12).

1. Shew that if one side of a rectangle be a measure of another, the figure can be divided into a number of equal squares.

2. Shew that if two adjacent sides of a rectangle have a common measure, the figure can be divided into a number of equal squares, each square having its sides equal to the common measure.

3. If the unit of length be 3 inches, shew that the unit of area is divisible into 9 one-inch squares.

4. If the length and breadth of a rectangle are 12 and 8 inches respectively, shew that the figure can be divided into 6 four-inch squares.

5. If two adjacent sides of a rectangle are $2\frac{1}{2}$ and $3\frac{1}{3}$ inches respectively, prove that its area = $(2\frac{1}{2} \times 3\frac{1}{3})$ square inches.

6. If two adjacent sides of a rectangle are 3.4 and 5.7 inches respectively, prove that its area = (3.4×5.7) square inches.

7. If two adjacent sides of a rectangle are 16 and 12 inches respectively, what is the *smallest* number of squares into which the figure can be divided?

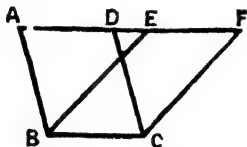
8. If the unit of length is doubled, the unit of area is quadrupled; if the unit of length is trebled, the unit of area is increased nine times. Illustrate this statement.

SECTION II.

THEOREMS.

Theorem 1. (Euc. I. 35.)

Parallelograms on the same base and between the same parallels are equal in area.



Let the par^m . ABCD and EBCF be on the same base BC and between the same parallels AF, BC.

To prove that the par^m . ABCD and EBCF are equal in area.

Proof. Since DC is \parallel to AB,

\therefore the $\angle FDC =$ the int. opp. $\angle EAB$.

Again, since EB is \parallel to FC,

\therefore the $\angle DFC =$ the ext. $\angle AEB$.

Thus, in the two Δ s FDC, EAB, we have

(1) the $\angle FDC =$ the $\angle EAB$,

(2) the $\angle DFC =$ the $\angle AEB$,

and (3) $DC = AB$; (*opp. sides of a par^m .*)

\therefore the two Δ s are equal in all respects.

(*Th. 18, Bk. I.*)

Hence, the whole quadl. ABCF *minus* the Δ FDC

$=$ the whole quadl. ABCF *minus* the Δ EAB.

(*Ax. 3.*)

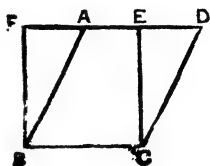
\therefore the par^m . ABCD $=$ the par^m . EBCF.

Q. E. D.

Cor. *The area of a parallelogram is equal to that of a rectangle contained by the base and altitude of the parallelogram.*

Let $ABCD$ be a par^m . on the base BC . Draw $CE \perp$ to AD ; then EC is the altitude of the par^m .

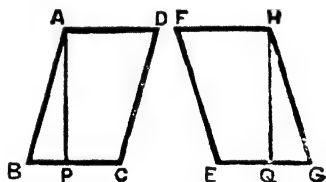
Draw $BF \parallel$ to CE , meeting DA produced in F . Then $FBCE$ is a rectangle, and it is contained by BC and CE .



Now, since FC and AC are par^m . on the same base and between the same parallels, \therefore they are equal in area; which proves the proposition.

Note. *Hence parallelograms on equal bases and of equal altitudes are equal in area.*

Let $ABCD$, $FEGH$ be two par^m . on equal bases BC and EG , and having equal altitudes AP and HQ .



Now, the par^m . $ABCD =$ the rect. BC, AP ; }
and the par^m . $FEGH =$ the rect. EF, HQ . }

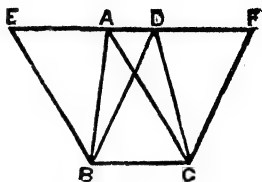
But $BC \cdot AP = EG \cdot HQ$;

\therefore the par^m . $ABCD =$ the par^m . $FEGH$.

Theorem 2. (Euc. I. 37.)

Triangles on the same base and between the same parallels are equal in area.

Let the \triangle s ABC, DBC be on the same base BC and between the same \parallel s AD, BC.



To prove that the \triangle s ABC, DBC are equal in area.

Draw BE \parallel to CA, meeting DA produced in E; and draw CF \parallel to BD, meeting AD produced in F.

Proof. EBCA and DBCF are par^{ms} ; and they are on the same base BC, and between the same \parallel s EF, BC.

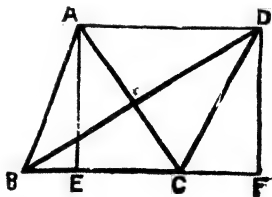
\therefore the par^{m} . EBCA = the par^{m} . DBCF.

But, the \triangle ABC = half the par^{m} . EBCA; } (Th. 25.
and the \triangle DBC = half the par^{m} . DBCF. } Bk. I.)

\therefore the \triangle ABC = the \triangle DBC. (Ax. 7.)
Q. E. D.

Cor. *Triangles on the same base and of the same altitude are equal in area.*

Let ABC and DBC be two \triangle s on the same base BC and having equal altitudes AE, DF.



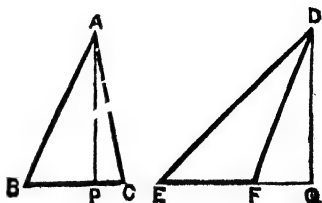
Now AE is \parallel to DF, and also equal to it: \therefore AD is \parallel to EF, i.e., to BC. (Th. 22, Bk. I.)

Hence the \triangle s ABC and DBC are in the same base BC and between the same \parallel s;

\therefore the \triangle ABC = the \triangle DBC.

Note 1. *Triangles on equal bases and of the same altitude are equal in area.*

Let ABC and DEF be two Δ s on equal bases BC and EF , and having equal altitudes AP , DQ .



Suppose the ΔDEF taken up and so placed that EF may coincide with BC , and D may fall on the same side of BC as A . Then the two Δ s will be on the same base and of the same altitude, and hence they are equal in area.

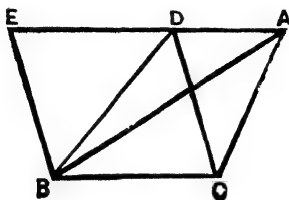
Note 2. Hence triangles on equal bases and *between the same parallels* are equal in area. For, triangles between the same parallels are of the same altitude.

Note 3. *Any median of a triangle divides the triangle into two equal parts.* For, the triangles on the two sides of the median are on equal bases and have a common altitude.

— — —

Theorem 3. (Euc. I. 41.),

If a triangle and a parallelogram be on the same base and between the same parallels, then the area of the triangle is equal to half that of the parallelogram.



Let the $\triangle ABC$ and the paral^m. BCDE be on the same base BC, and between the same \parallel s EA, BC.

To prove that the $\triangle ABC$ is half of the paral^m. BCDE.

Proof. Join BD.

The diagonal BD bisects the paral^m. EC ;

\therefore the $\triangle DBC$ is half of the paral^m. BCDE.

But the $\triangle DBC =$ the $\triangle ABC$; because they are on the same base BC, and between the same \parallel s BC, DA ;

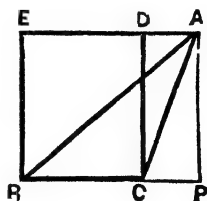
\therefore the $\triangle ABC$ is half of the paral^m. BCDE.

Q. E. D.

Cor. The area of a triangle is equal to half that of a rectangle contained by the base and altitude of the triangle.

Let ABC be a \triangle of which BC is the base and AP, the altitude.

Draw AE \parallel to PB ; also draw CD and BE each \parallel to PA, meeting AE in D and E respectively.



Then the par^m . EC and the $\triangle ABC$ are on the same base and between the same \parallel s, \therefore the $\triangle ABC = \frac{1}{2}$ of the par^m . EC.

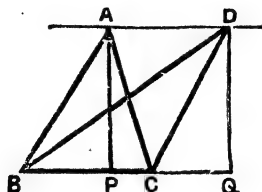
But, since CD is \parallel to PA, the par^m . EC is evidently a *rectangle*; and it is contained by BC and CD, *i.e.*, by BC and PA.

Hence the $\triangle ABC = \frac{1}{2}$ the rect. contained by BC and AP.

Note. Hence if the base and altitude of a triangle are respectively equal to b and h units of length, then the area of the $\triangle = \frac{1}{2}bh$ units of area.

Theorem 4.

Equal triangles on the same base are of the same altitude,



Let the \triangle s ABC and DBC, standing on the same base BC, be equal in area.

Let AP and DQ be the altitudes of the \triangle s ABC and DBC.

To prove that $AP = DQ$.

Proof. The $\triangle ABC = \frac{1}{2}$ the rect. BC, AP ; } (Th. 3,
and the $\triangle DBC = \frac{1}{2}$ the rect. BC, DQ. } Cor.)

\therefore the rect. BC, AP = the rect. BC, DQ.

$\therefore AP = DQ$. Q. E. D.

Note. It may be similarly proved that *equal triangles on equal bases are of the same altitude.*

Cor. *Equal triangles on the same base (or on equal bases in the same straight line) and on the same side of it are between the same parallels.*

For in the above diagram, AP being equal and \parallel to DQ, AD is \parallel to PQ, i.e., to BC.

EXERCISE (13).

1. Prove that the four triangles into which a parallelogram is divided by its diagonals are equal in area.
2. Prove that equal triangles on opposite sides of the same base have the same altitude. Hence prove that if

two equal triangles ABC , DBC be on opposite sides of the same base BC , AD is bisected by BC or BC produced.

3. BC and AD are the parallel sides of a trapezium and E is the mid-point of CD . Prove that the $\triangle AEB$ is half of the whole figure.

4. If two triangles have two sides of the one equal to two sides of the other, and the contained angles supplementary; shew that the two triangles can be so placed as to form two parts of the same triangle, and hence that they are equal in area.

5. If two sides of a triangle are given, prove that the area is greatest when the included angle is a right angle.

6. Prove that if a quadrilateral be bisected by each of its diagonals, it must be a parallelogram.

7. Prove that in any quadrilateral the straight lines joining the middle points of opposite sides bisect each other.

8. If E , F be the mid-points of the sides AC , AB of a triangle ABC , prove that the triangle AEF is one-fourth of the whole triangle ABC .

9. D , E are the mid-points of the sides AB , AC of a triangle ABC ; and P is any point in BC produced. Prove that the triangle PDE is one-fourth of the triangle ABC .

10. Find the area of a triangle whose base and altitude are 3.5 and 2.8 inches respectively.

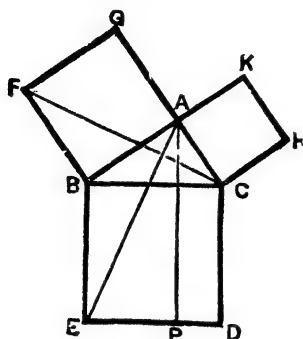
11. Find the altitude of a triangle whose area is 3.75 sq. inches and the base, 3 inches.

12. Find the base of a triangle whose area is 19.25 sq. inches and the altitude, 7 inches.

13. If two parallelograms have two adjacent sides of the one respectively equal to two adjacent sides of the other, and the included angles supplementary, prove that the parallelograms are equal in area.

Theorem 5. (Euc. I. 47.)

In a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the sides containing the right angle.



Let ABC be a right-angled \triangle , having the right angle at A . Let $BCDE$, $ABFG$ and $ACHK$ be the squares described on BC , AB and AC respectively.

To prove that the sq. BD = the sum of the squares AF and AH .

Draw $AP \parallel$ to BE or CD , meeting DE in P .

Join AE , CF .

Proof. Since each of the adj. \angle s BAC , BAG is a rt. \angle ;

\therefore CA and AG are in one str. line.

Now, the $\angle CBE$ = the $\angle ABF$; (each being a rt. \angle).

\therefore adding the $\angle ABC$ to each of them, we have the whole $\angle ABE$ = the whole $\angle FBC$.

Then in the two \triangle s ABE , FBC , we have

$$(1) \quad AB = FB,$$

$$(2) \quad BE = BC,$$

and (3) the $\angle ABE$ = the $\angle FBC$:

\therefore the two \triangle s are equal in all respects. (*Th. 4, Bk. I.*)

Now, the $\triangle ABE$ and the rect. BP are on the same base BE and between the same \parallel s AP, BE ;

\therefore the rect. $BP =$ twice the $\triangle ABE$. (Th. 3.)

Also, since the $\triangle FBC$ and the sq. AF are on the same base FB and between the same \parallel s GC, FB ,

\therefore the sq. $AF =$ twice the $\triangle FBC$. (Th. 3.)

Hence the rect. $BP =$ the sq. AF . (Ax. 6.)

Similarly, by joining AD and BH , it may be shewn that the rect. $CP =$ the sq. AH .

\therefore the rectangles BP and CP are together

$=$ the sum of the squares AF and AH ;

i.e., the sq. $BD =$ the sum of the sqs. AF and AH .

Q. E. D.

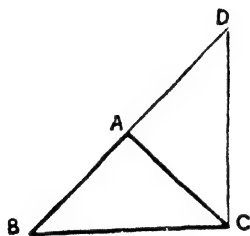
Note 1. The above result may be briefly written as follows ;
 $BC^2 = CA^2 + AB^2$.

Note 2. If the sides BC, CA, AB are respectively equal to a, b, c , units of length, then

$$\left. \begin{array}{l} \text{Hence} \quad a^2 = b^2 + c^2, \\ \text{and} \quad b^2 = a^2 - c^2, \\ \quad \quad c^2 = a^2 - b^2. \end{array} \right\}$$

Theorem 6. (Euc. I. 48.)

If the square described on one side of a triangle be equal to the sum of the squares described on the other two sides, then the angle contained by those two sides is a right angle.



Let ABC be a triangle such that $BC^2 = CA^2 + AB^2$.

To prove that the $\angle BAC$ is a right angle.

Draw $AD \perp$ to BC , and make $AD = AB$.

Join DC .

Proof. Since the $\angle CAD$ is a rt. \angle ,

$$\begin{aligned} \therefore DC^2 &= CA^2 + AD^2 && (Th. 5.) \\ &= CA^2 + AB^2. \end{aligned}$$

Hence the sq. on DC = the sq. on BC ;

$$\therefore DC = BC.$$

Now, in the two \triangle s ABC , ADC , we have

$$(1) \quad AB = AD, \quad (Cons.)$$

$$(2) \quad AC \text{ common,}$$

and (3) $BC = DC$,

\therefore the two \triangle s are congruent. (Th. 7, Bk. I.)

Hence the $\angle BAC =$ the $\angle DAC =$ a rt. \angle . Q. E. D.

Note 1. We have $3^2 + 4^2 = 5^2$. Hence if the sides of a triangle are 3, 4 and 5 units of length respectively, the triangle is right-angled.

Note 2. Again, since $5^2 + 12^2 = 13^2$, \therefore the triangle whose sides are 5, 12 and 13 units of length respectively, is right-angled.

Note 3. We have $(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$. Hence, generally speaking, if the sides of a triangle are $(m^2 - n^2)$, $2mn$ and $(m^2 + n^2)$ units of length respectively, the triangle is right-angled.

Hence for each pair of numbers that may be substituted for m and n , we shall get a set of three numbers representing three str. lines which can form a right-angled triangle. Thus if $m=2$ and $n=1$, we have the sides equal to 3, 4 and 5 units respectively; if $m=4$ and $n=3$, we have the sides equal to 7, 24 and 25 units respectively; if $m=4$ and $n=1$, we have the sides equal to 15, 8 and 17 units respectively; and so on.

EXERCISE (14).

1. Construct a square whose area is double that of a given square.

2. Construct a square whose area is treble that of a given square. Hence shew how to construct a square whose area is seven times that of a given square.

3. A and B are two fixed points; and CD is any straight line perpendicular to AB, meeting AB or AD produced in D. If P be *any* point in CD, prove that the difference of the squares on PA and PB is equal to the difference of the squares on DA and DB.

4. Prove that a triangle whose sides are respectively 28, 45 and 53 units of length, is a right-angled triangle.

5. If a man starting from a place A travels 12 miles due North and then 5 miles due East, how far is he from A at the end of his journey?

6. A ladder 37 feet long rests with one end on the ground and the other on the top of a wall. Calculate the height of the wall if the distance of the foot of the ladder from the wall be 12 feet.

7. A rod 13 feet long is held vertically in a tank of water, one end of the rod reaching the bottom of the tank, and the other being *above* the surface of the water. Another rod of equal length is held in a slant position, one end being in contact with the lower end of the first rod and the other being *on the surface* of the water. If the upper end of the second rod be 5 feet distant from the point where the first cuts the surface of water, calculate the length of that portion of the first rod which is above the water.

8. Divide a given straight line into two parts so that the sum of the squares on the parts may be equal to a given square.

9. Divide a given straight line into two parts so that the difference of the squares on the parts may be equal to a given square.

10. Find a straight line the square on which is equal to the difference of two given squares.

11. AD is the altitude of an equilateral triangle ABC. If AD and BD respectively contain p and m units of length, prove that $p^2 = 3m^2$.

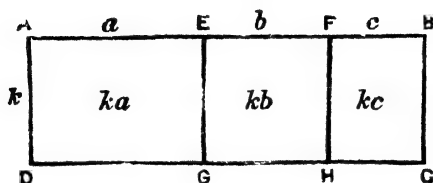
12. ABC is a triangle acute-angled at B. Prove that AC^2 is less than $AB^2 + BC^2$.

Hence prove that if in a triangle ABC, $AC^2 > AB^2 + BC^2$, then the angle ABC is obtuse.

Theorem 7. (Euc. II. 1.)

To illustrate and explain the geometrical theorem corresponding to the identity

$$k(a + b + c) = ka + kb + kc.$$



Let ABCD be a rectangle of which the side AB is divided into any number of parts AE, EF and FB.

Suppose that AD, AE, EF and FB are respectively equal to k , a , b and c units of length.

Draw EG and FH \parallel to AD, meeting DC in G and H.

Then the figures AG, EH and FC are rectangles and their areas are respectively equal to ka , kb and kc units of area.

And the area of the whole rectangle AC
 $= k(a + b + c)$ units of area.

But the whole rectangle = the sum of its parts ;

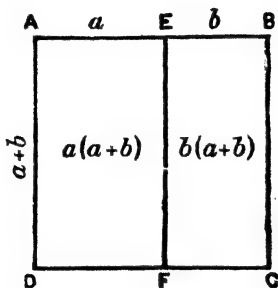
$$\therefore k(a + b + c) = ka + kb + kc.$$

Thus, if there are two straight lines of which one is divided into any number of parts, the rectangle contained by the two lines is equal to the sum of the rectangles contained by the undivided line and the several parts of the divided line.

Theorem 8. (Euc. II. 2.)

To illustrate and explain the geometrical theorem corresponding to the identity.

$$(a+b)^2 = a(a+b) + b(a+b).$$



Let ABCD be a square of which the side AB is divided into any two parts AE and EB.

Suppose that AE and EB are respectively equal to a and b units of length ; then the side AD = $(a+b)$ units of length.

Draw EF \parallel to AD, meeting DC in F.

Then each of the figures AF and EC is a rectangle ; and their areas are respectively equal to $a(a+b)$ and $b(a+b)$ units of area.

And the area of the whole figure AC
 $= (a+b)^2$ units of area.

But the whole figure is equal to the sum of its parts ;

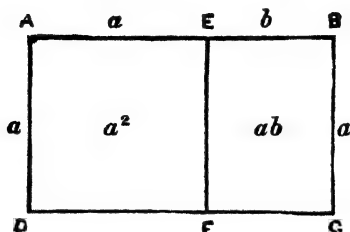
$$\therefore (a+b)^2 = a(a+b) + b(a+b).$$

Thus, *if a straight line be divided into any two parts, the square on the straight line is equal to the sum of the rectangles contained by the whole line and each of the parts*

Th orem 9. (EUC. II. 3.)

To illustrate and explain the geometrical theorem corresponding to the identity

$$a(a+b) = a^2 + ab.$$



Let ABCD be a rectangle, of which the side AB is divided into two parts AE and EB, such that $AE = AD$.

Suppose that AE and EB are respectively equal to a and b units of length.

Draw EF \parallel to AD, meeting DC in F.

Then the figure AF is a square, and the figure EC is a rectangle ; their areas being respectively equal to a^2 and ab units of area.

And the area of the whole figure AC

$$= a(a+b) \text{ units of area.}$$

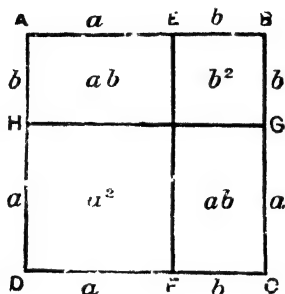
But the whole figure is equal to the sum of its parts ;

$$\therefore a(a+b) = a^2 + ab.$$

Thus, if a straight line be divided into any two parts, the rectangle contained by the whole line and one of the parts is equal to the square on that part together with the rectangle contained by the two parts.

Theorem 10. (Euc. II. 4.)

To illustrate and explain the geometrical theorem corresponding to the identity

$$(a+b)^2 = a^2 + 2ab + b^2.$$


Let ABCD be a square of which the side AB is divided into any two parts AE and EB; and suppose that these two parts are respectively equal to a and b units of length.

Draw EF \parallel to AD, meeting DC in F. From BC cut off BG = BE, and draw GH \parallel to BA meeting AD in H.

Then the figures HE and FG are rectangles, and the area of each of them = ab units of area; and the figures EG, FH are squares, their areas being respectively equal to b^2 and a^2 units of area.

Also the area of the whole figure AC
 $= (a+b)^2$ units of area.

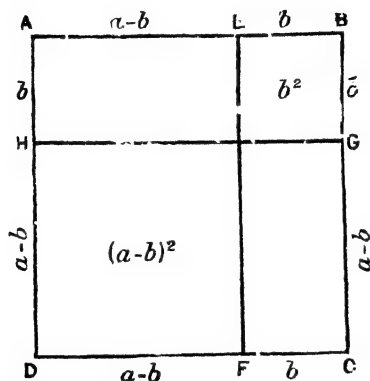
But the whole figure is equal to the sum of its parts;
 $\therefore (a+b)^2 = a^2 + b^2 + 2ab.$

Thus, *if a straight line be divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts together with twice the rectangle contained by the parts.*

Theorem 11. (Euc. II. 7.)

To illustrate and explain the geometrical theorem corresponding to the identity

$$(a - b)^2 = a^2 - 2ab + b^2.$$



Let ABCD be a square of which the side AB is divided into two parts AE and EB.

Suppose that the whole str. line AB and its part BE are respectively equal to a and b units of length ; then the part AE = $(a - b)$ units of length.

Draw EF \parallel to AD, meeting DC in F; cut off BG = BE, and draw GH \parallel to BA, meeting AD in H,

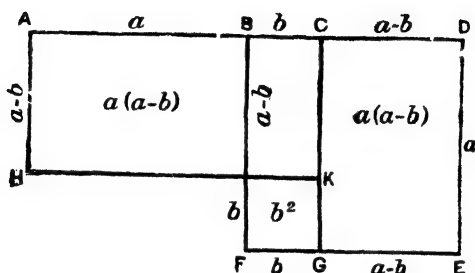
Then the areas of the rectangles AG, EC are each = ab units of area ; and the areas of the squares HF and EG are respectively = $(a - b)^2$ and b^2 units of area.

Also the area of the whole square AC = a^2 units of area.

Theorem 12. (EUC. II. 5 and 6.)

To illustrate and explain the geometrical theorem corresponding to the identity.

$$a^2 - b^2 = (a + b)(a - b).$$



Let the two str. lines AB and BC be in the same str. line, and let them contain a and b units of length respectively.

Produce BC to D making $BD = AB$; then $CD = (a - b)$ units of length.

Let BDEF be the square on BD ;

draw $CG \parallel$ to BF, meeting FE in G ;

draw $AH \parallel$ to BF, making $AH = CD$;

and draw $HK \parallel$ to AC, meeting CG in K.

The rectangles HB and CE are equal ; the area of the rect. AK = $(a + b)(a - b)$ units of area ; and the areas of the squares BE and FK are respectively = a^2 and b^2 units of area.

Now, the difference of the two squares

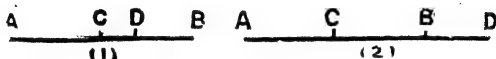
= the rect. CE + the rect. BK

$$\begin{aligned}
 &= \text{the rect. HB} + \text{the rect. BK} \\
 &= \text{the rect. AK.}
 \end{aligned}$$

Hence $a^2 - b^2 = (a + b)(a - b)$.

Thus, *the difference of the squares on any two straight lines is equal to the rectangle contained by their sum and difference.*

Note. *If a straight line be bisected, and also divided into two unequal segments (either internally or externally), then the rectangle contained by the unequal segments is equal to the difference of the squares on half the line and on the line between the points of section.*



For, if the str. line **AB** be bisected at **C** and also divided into two unequal segments at **D**, then in fig. (1) we have

$$\begin{aligned}
 \text{AD. DB} &= (\text{AC} + \text{CD})(\text{CB} - \text{CD}) \\
 &= (\text{AC} + \text{CD})(\text{AC} - \text{CD}) \\
 &= \text{AC}^2 - \text{CD}^2;
 \end{aligned}$$

and in fig. (2)

$$\begin{aligned}
 \text{AD. DB} &= (\text{CD} + \text{AC})(\text{CD} - \text{CB}) \\
 &= (\text{CD} + \text{AC})(\text{CD} - \text{AC}) \\
 &= \text{CD}^2 - \text{AC}^2.
 \end{aligned}$$

EXERCISE (15).

1. Illustrate and explain the geometrical theorem corresponding to the identity

$$(2a)^2 = 4a^2.$$

2. Illustrate and explain the geometrical theorem corresponding to the identity

$$(3a)^2 = 9a^2.$$

3. Prove geometrically that

$$(a + b)(c + d) = ac + ad + bc + bd.$$

4. Prove geometrically that

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca.$$

5. Prove geometrically that

$$(a - b)(a - c) = a^2 - ac + bc - ab.$$

6. Illustrate and explain the geometrical theorem corresponding to the identity

$$(a + b)(b + c) = ac + b(a + b + c).$$

7. If AD is the perpendicular upon the hypotenuse BC of a right-angled triangle ABC, prove that

$$AD^2 = BD \cdot DC.$$

Hence prove that (i) $AB^2 = BC \cdot BD$, and (ii) $AC^2 = BC \cdot CD$.

8. ABC is a triangle, and AD is perpendicular to BC. If $AD^2 = BD \cdot DC$, prove that BAC is a right angle.

9. C is any point in a given straight line AB. Deduce from Theorems 9 and 10, that

$$AB^2 + BC^2 = AC^2 + 2AB \cdot BC.$$

10. If C is the mid-point of a given straight line AB, and if D is any point in CB, prove that

$$AD^2 + DB^2 = 2AC^2 + 2CD^2.$$

11. If C is the mid-point of a given straight line AB, and if D is any point in CB produced, prove that

$$AD^2 + DB^2 = 2AC^2 + 2CD^2.$$

12. C is the mid-point of a given straight line AB. If a point D moves from B to A, prove that the area of the rect. AD. DB gradually *increases* until D coincides with C, when it has its *greatest* value, and thereafter gradually *diminishes*.

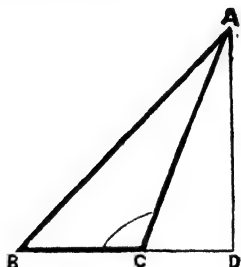
13. In the previous case, prove that $AD^2 + DB^2$ gradually *diminishes* until B coincides with C, when it has its *least* value, and thereafter gradually *increases*.

Hence shew that if the sum of two straight lines be given, the least value of the sum of the squares on the lines is equal to half the square on the given sum.

14. AB is a given straight line. If a point C moves from B to A, prove that $AB^2 + BC^2$ is always *greater than* $2AB \cdot BC$, except when C coincides with A; and that in this case $AB^2 + BC^2 = 2AB \cdot BC$.

Theorem 13. (EUC. II. 12.)

In an obtuse-angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the other two sides together with twice the rectangle contained by one of these two sides and the projection on it of the other.



Let ABC be a \triangle obtuse-angled at C ; and let AD be \perp to BC produced, so that CD is the projection of AC on BC.

To prove that

$$AB^2 = BC^2 + CA^2 + 2BC \cdot CD.$$

Proof. $BD^2 = BC^2 + CD^2 + 2BC \cdot CD.$ (Th. 10.)

To each of these equals add DA^2 .

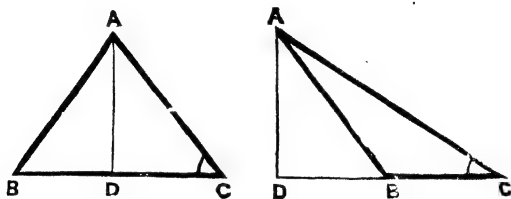
$$\text{Then } BD^2 + DA^2 = BC^2 + CD^2 + DA^2 + 2BC \cdot CD.$$

$$\left. \begin{array}{l} \text{But } BD^2 + DA^2 = AB^2, \\ \text{and } CD^2 + DA^2 = AC^2. \end{array} \right\} \quad (\text{Th. 5.})$$

$$\therefore AB^2 = BC^2 + CA^2 + 2BC \cdot CD. \quad \text{Q. E. D.}$$

Theorem. 14. (Euc. II. 13.)

In any triangle the square on the side opposite to an acute angle is equal to the sum of the squares on the other two sides diminished by twice the rectangle contained by one of these two sides and the projection on it of the other.



Let ABC be any \triangle acute-angled at C ; and let AD be \perp to CB or CB produced, so that CD is the projection of AC on BC .

To prove that

$$AB^2 = BC^2 + CA^2 - 2BC \cdot CD.$$

Proof. Since BD is the difference of the two str. lines BC, CD , we have

$$BD^2 = BC^2 + CD^2 - 2BC \cdot CD. \quad (Th. 11.)$$

To each of these equals add DA^2 .

$$\text{Then } BD^2 + DA^2 = BC^2 + CD^2 + DA^2 - 2BC \cdot CD.$$

$$\text{But } BD^2 + DA^2 = AB^2, \quad (Th. 5.)$$

$$\text{and } CD^2 + DA^2 = AC^2.$$

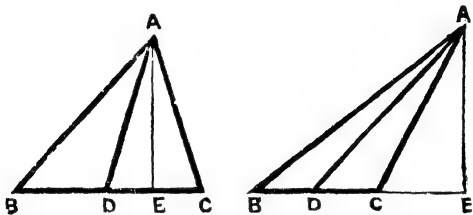
$$\therefore AB^2 = BC^2 + CA^2 - 2BC \cdot CD. \quad Q. E. D.$$

Note. Comparing Theorems 5, 13 and 14, we come to the following conclusion :—

The square on a side of a triangle is greater than, equal to, or less than the sum of the squares on the other two sides, according as the angle contained by those sides is obtuse, right, or acute; the difference in the cases of inequality being twice the rectangle contained by one of the two sides and the projection on it of the other.

Theorem 15.

The sum of the squares on any two sides of a triangle is equal to twice the square on half the third side together with twice the square on the median that bisects the third side.



Let AD be the median which bisects the side BC of a $\triangle ABC$.

To prove that $AB^2 + AC^2 = 2BD^2 + 2DA^2$.

Proof. Let AE be \perp to BC or BC produced; then in the above diagram, the $\angle ADB$ is obtuse and \therefore the $\angle ADC$ is acute.

Hence in the $\triangle ABD$,

$$AB^2 = BD^2 + DA^2 + 2BD \cdot DE; \quad (\text{Th. 13.})$$

and in the $\triangle ADC$,

$$\begin{aligned} AC^2 &= CD^2 + DA^2 - 2CD \cdot DE \\ &= BD^2 + DA^2 - 2BD \cdot DE. \end{aligned} \quad (\text{Th. 14.})$$

Hence by addition.

$$AB^2 + AC^2 = 2BD^2 + 2DA^2. \quad \text{Q. E. D.}$$

EXERCISE (16).

1. The numerical measures of the sides of a triangle are 7, 9 and 11: prove that the angle opposite to the greatest side is neither right nor obtuse.

Construct the triangle, taking one-tenth of an inch as the unit of length.

2. The numerical measures of the sides of a triangle are 8, 13 and 17 : shew that the angle, opposite to the greatest side must be obtuse.

Construct the triangle, taking three-tenths of a centimetre as the unit of length.

3. ABC is an acute-angled triangle. The perpendiculars from A, B, C upon the opposite sides meet at O. Prove that

$$OA^2 + OB^2 + OC^2 < \frac{1}{2}(AB^2 + BC^2 + CA^2).$$

4. The numerical measure of a side of an equilateral triangle ABC is a . BC is produced to D so that $CD = 2BC$. If x be the numerical measure of AD, prove that $x^2 = 7a^2$.

5. ABC is an acute-angled triangle; and BE, CF are perpendiculars upon CA, AB. Prove that $AB \cdot AF = AC \cdot AE$.

6. D is the mid-point of the side BC of a triangle ABC; and AE is perpendicular to BC, or BC produced. Use Theorems 13 and 14 to prove that the difference of the squares on AB, AC is equal to twice the rect. BC.DE.

7. C is the mid-point of a given straight line AB. If a point P moves on the circumference of a circle of which the centre is C, prove that $PA^2 + PB^2$ is always the same.

8. The distance between two fixed points A and B is 6 inches. If a point P moves so that $PA^2 + PB^2$ is always = 50 square inches, find the locus of P.

9. Prove that the sum of the squares on the two equal sides of an isosceles triangle is less than the sum of the squares on the two sides of any other triangle on the same base and between the same parallels.

10. ABCD is a rectangle ; and P is any point, either inside or outside it. Prove that $PA^2 + PC^2 = PB^2 + PD^2$.

11. Prove that the sum of the squares on the sides of a parallelogram is equal to the sum of the squares on its diagonals.

12. Prove that three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on the medians.

13. If O is the centroid of a $\triangle ABC$, prove that $3(OA^2 + OB^2 + OC^2) = AB^2 + BC^2 + CA^2$.

14. Prove that the sum of the squares on the sides of a quadrilateral is equal to the sum of the squares on its diagonals together with four times the square on the line joining the middle points of the diagonals.

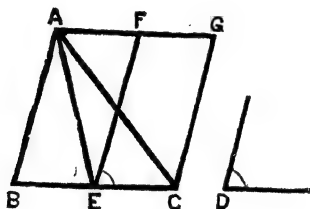
Hence prove that if the sum of the squares on the sides of a quadrilateral is equal to the sum of the squares on the diagonals, the quadrilateral must be a parallelogram.

SECTION III.

PROBLEMS.

Problem 1. (EUC. I. 42.)

To construct a parallelogram equal to a given triangle and having one of its angles equal to a given angle.



Let ABC be a given \triangle , and D the given \angle .

It is required to construct a par^m. equal to the $\triangle ABC$, and having one of its \angle s = the $\angle D$.

Cons. Bisect BC at E .

At E in CE , make the $\angle CEF = \angle D$; draw $CG \parallel$ to EF ; draw $AG \parallel$ to BC , meeting EF and CG in F and G respectively.

Then the figure FC is the required par^m.

Proof. Join AE .

Since the two \triangle s ABE , AEC are on equal bases BE , EC , and of the same altitude;

\therefore the $\triangle ABE =$ the $\triangle AEC$.

\therefore the $\triangle ABC$ is double of the $\triangle AEC$ (a)

The quadl. FC is a par^m., because its opposite sides are \parallel .

Now, the par^m. FC and the $\triangle AEC$ are on the same base EC and between the same \parallel s AG, EC ;

\therefore the par^m. FC is double of the $\triangle AEC$ (b)

Hence, from (a) and (b),

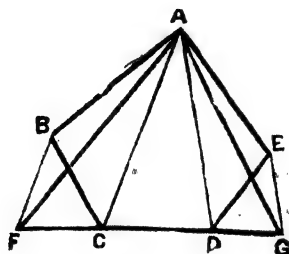
the par^m. $FC =$ the $\triangle ABC$;

and it has also the $\angle CEF =$ the $\angle D$. Q. E. F.

Note. If the given $\angle D$ be a right angle, the parallelogram constructed will be a *rectangle*.

Problem 2.

To construct a triangle equal in area to any given rectilineal figure.



Let $ABCDE$ be a given rectilineal figure.

It is required to construct a \triangle equal in area to the rectil. figure $ABCDE$.

Join AC , AD .

Produce DC to F , where AC meets DF ; and produce CD to G , where AD meets CG .

Join AG .

The required \triangle is AFG .

BC , AFC are on the same base AC , and between the same parallels AC , BF ;

$\therefore \triangle AFC = \triangle BFC$... (a)

$\therefore \triangle AFG$ is on the same base AD as $\triangle ADE$

... (b)

Hence, from (a) and (β),

$$\begin{aligned} & \text{the } \triangle AFC + \text{the } \triangle ACD + \text{the } \triangle AGD \\ & = \text{the } \triangle ABC + \text{the } \triangle ACD + \text{the } \triangle AED ; \end{aligned}$$

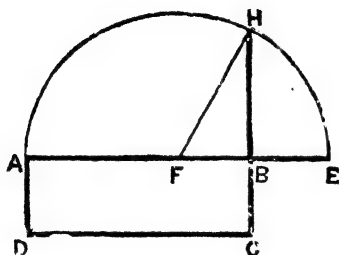
i.e., the $\triangle AFG =$ the rectil. figure ABCDE. Q. E. F.

Note 1. The given rectil. figure is = the rectil. figure ABCG which has one side less than those of the given figure. From this it is clear that a *seven* sided rectil. figure can be reduced to one of *six* sides, and this latter again to one of *five* sides ; and so on. Thus, by successive repetitions of the above process, any given rectil. figure may be finally reduced to a triangle of equal area.

Note 2. We can construct a *rectangle* equal to any given rectil. figure. For, all that we have to do is to reduce the given figure to a triangle of equal area, and then construct a rectangle equal to this triangle.

Problem 3.

To construct a square equal in area to a given rectangle.



Let ABCD be the given rectangle, of which the side AB is $>$ the side BC.

It is required to construct a square
to the rectangle AC.

Produce AB to E, making $BE = BC$.

mid-pt. of AE ; with centre F, and radius
a semi-circle on AE.

+ the \bigcirc^{re} in H.

is the required square.

rt. \angle .

(Th. 5.)

(Th. 12.)

Q. F.

Note. We can now construct a square equal to any given rectil. figure. For, all that we have to do is to construct a *rectangle* equal to the given figure (Note 2, Prob. 2), and then construct a square equal to this rectangle.

EXERCISE (17).

1. On one side of a triangle construct a parallelogram equal to the triangle.

2. On one side of a triangle as base construct an isosceles triangle, equal to half the given triangle.

Hence shew how to construct a *rhombus* equal to a given triangle, having one side of the triangle as a diagonal.

3. Bisect a parallelogram by a straight line drawn through any given point.

4. Bisect a triangle by a straight line drawn through a given point in one of its sides.

5. Bisect a quadrilateral by a straight line drawn through one of its angular points.

6. P is a given point in the side AB of a triangle ABC. Find a point D in BC produced so that the triangle PBD may be equal to the triangle ABC.

7. ABC is a triangle and P is a point in BA produced. Find a point D in BC so that the triangle PBD may be equal to the triangle ABC.

8. Divide a triangle into five equal parts by straight lines drawn through one of its angles.

Hence cut off from a triangle a part equal to it by a straight line drawn through one of its sides.

9. Trisect a triangle by straight lines drawn through a given point in one of its sides.

10. On a given straight line AB construct a rectangle equal to a given square.

11. On a given straight line AB construct a rectangle equal to a given rectangle.

12. On a given straight line construct a rectangle equal to a given triangle.

13. On a given straight line construct a triangle equal to a given triangle.

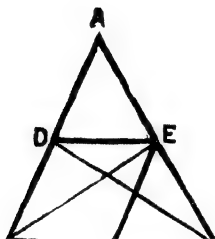
14. Shew that if a rectangle be equal to a square, the perimeter of the square is less than that of the rectangle.

Hence shew that of all rectangles having the same area the square is that which has the *least* perimeter.

SECTION IV.

MISCELLANEOUS PROPOSITIONS.

1. *The straight line joining the middle points of any two sides of a triangle is parallel to the third side, and equal to half of it.*



Let D, E be the mid-pt. of the sides AB and AC of a $\triangle ABC$.

To prove that DE is \parallel to BC and is $= \frac{1}{2} BC$.

Proof. (1) Join BE, CD.

The \triangle s BDC, ADC are on equal bases and of the same altitude ;

\therefore the $\triangle BDC =$ the $\triangle ADC$;

and \therefore the $\triangle BDC = \frac{1}{2}$ the $\triangle ABC$. }

Similarly, the $\triangle BEC = \frac{1}{2}$ the $\triangle ABC$. }

Hence the $\triangle BDC =$ the $\triangle BEC$.

\therefore DE is \parallel to BC. (Cor. Th. 4.)

(2) Let F be the mid-pt. of BC; join EF.

Then since E and F are the mid-pt. of AC and BC,

\therefore EF is \parallel to AB.

Hence DF is a par^m;

$\therefore DE = BF = \frac{1}{2} BC$.

Q. E. D.

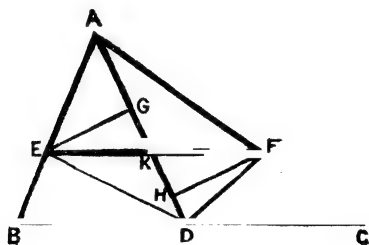
Cor. *Conversely, if through the middle point of a side of a triangle a straight line be drawn parallel to another, it will pass through the middle point of the third side.*

In the above diagram, if $DE \parallel BC$, then the $\triangle BEC =$ the $\triangle BDC$.

\therefore the $\triangle BEC = \frac{1}{2}$ the $\triangle ABC$.

Hence the pt. E is the mid-pt. of AC (for, if any other pt. E' were the mid-pt., the $\triangle BE'C$ would be half of the $\triangle ABC$, which is impossible).

2. *If D is the mid-pt. of the side BC of a triangle ABC , prove that AD bisects any straight line EF drawn parallel to BC meeting the sides AB , AC at E and F respectively.*



Proof. (i) Join DE , DF .

The $\triangle s$ BED , DFC are on equal bases BD , DC and between the same $\parallel s$ EF , BC ;

\therefore the $\triangle BED =$ the DFC . (Note 2, Th. 2.)

Also the $\triangle ABD =$ the $\triangle ADC$.

\therefore the $\triangle AED =$ the $\triangle AFD$. (Ax. 3.)

(ii) Let EG and FH be the altitudes of the $\triangle s$ AED , AFD ; and let K be the pt. where EF cuts AD .

The $\triangle AED = \frac{1}{2}$ the rect. $AD \cdot EG$; } (Cor. Th. 3.)
and the $\triangle AFD = \frac{1}{2}$ the rect. $AD \cdot FH$.

Hence $EG = FH$.

Now in the \triangle s EKG , FKH , we have

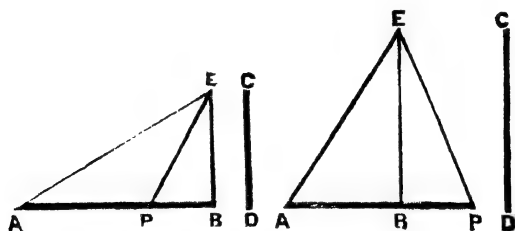
(1) the $\angle EKG =$ the $\angle FKH$,

(2) the $\angle EGK =$ the $\angle FHK$,

and (3) $EG = FH$.

Hence $EK = KF$. Q. E. D.

3. *AB is a given straight line. Find a point P in AB, or AB produced, such that the difference of the squares on AP, BP may be equal to the square on a given line CD.*



Cons. Draw $BE \perp$ to AB , making $BE = CD$.

Join AE ; and make the $\angle AEP = \angle EAB$.

Let EP meet AB , or AB produced, in P .

The point P thus found is the required point.

Proof. Since the $\angle AEP =$ the $\angle EAB$,

$$\therefore AP = EP.$$

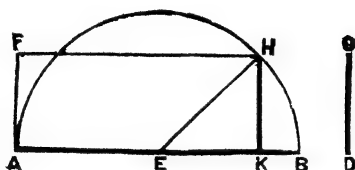
$$\text{Hence, } AP^2 - BP^2 = EP^2 - BP^2$$

$$= EB^2 \quad (\because \text{the } \angle EBP \text{ is right})$$

$$= CD^2.$$

Q. E. F.

4. *Divide a given straight line internally so that the rectangle contained by the two segments may be equal to a given square ; a side of the square being less than half the given straight line.*



Let AB be the given straight line, and CD a side of the given square.

To find a pt. K in AB so that the rect. $AK.KB$ may be = the sq. on CD .

Cons. Bisect AB at E .

With E as centre and EA as radius, describe a semi-circle on AB .

Draw $AF \perp$ to AB , making $AF = CD$.

Draw $FH \parallel$ to AB , and let H be one of the points where FH cuts the semi-circle.

Draw $HK \perp$ to AB ; then K is the reqd. pt.

Proof. Join EH .

FK is a rectangle ; $\therefore HK = FA = CD$.

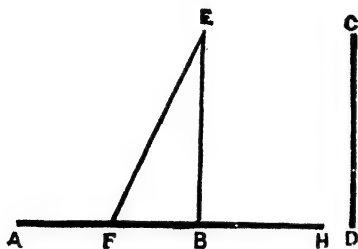
$$\begin{aligned}
 \text{Now, } AK.KB &= AE^2 - EK^2 && (\text{Th. 12, Note.}) \\
 &= EH^2 - EK^2 \\
 &= HK^2 \\
 &= CD^2.
 \end{aligned}$$

Q. E. F.

Note. Hence, given the area of a rectangle and the *sum* of its adjacent sides, we know how to construct the rectangle.

If the area be given as equal to that of a given *rectangle*, we have first of all to construct a square equal to this rectangle (Problem 3), and then proceed as above.

5. *Divide a given straight line externally so that the rectangle contained by the two segments may be equal to a given square.*



Let AB be the given str. line, and CD a side of the given square.

To find a pt. H in AB produced so that the rect. $AH \cdot HB$ may be = the sq. on CD .

Cons. Draw $BE \perp$ to AB , making $BE = CD$.

Bisect AB at F , and join FE .

Produce FB to H , making $FH = FE$.

Then H is the required pt.

Proof. AB is bisected at F and divided *externally* at H ;

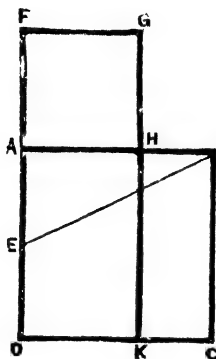
$$\begin{aligned}
 \therefore AH \cdot HB &= FH^2 - FB^2 && (\text{Th. 12, Note.}) \\
 &= FE^2 - FB^2 \\
 &= EB^2 \\
 &= CD^2.
 \end{aligned}$$

Q. E. F.

Note. Hence, given the area of a rectangle and the *difference* of its adjacent sides, we know how to construct the rectangle.

If the area be given as equal to that of a given rectangle, we have first of all to construct a square equal to this rectangle (Problem 3), and then proceed as above.

6. *AB is a given straight line. Find a point H in AB, so that the rectangle contained by AB and BH may be equal to the square on AH.*



Cons. On AB describe the square ABCD.

Bisect AD at E, and join EB.

Produce EA to F, making $EF = EB$.

From AB cut off $AH = AF$; then H is the reqd. pt.

Proof. Through H draw $GHK \parallel$ to FD, meeting DC in K; and draw $FG \parallel$ to AB, meeting KG in G.

Now FK is a rectangle, which is contained by DF, FG and \therefore by DF, FA; and FH is the sq. on AH.

Since DA is bisected at E and divided *externally* at F,

$$\begin{aligned} \therefore DF \cdot FA &= EF^2 - EA^2 && (\text{Th. 12, Note.}) \\ &= EB^2 - EA^2 \\ &= AB^2 \quad (\because \angle EAB \text{ is right}). \end{aligned}$$

Thus, the rect. FK = the sq. AC.

Taking away the common part AK, we have

$$\begin{aligned} \text{the sq. FH} &= \text{the rect. HC} \\ &= \text{the rect. AB, EH} \quad (\because CB = AB). \end{aligned}$$

That is, $AH^2 = AB \cdot BH$.

Q. E. F.

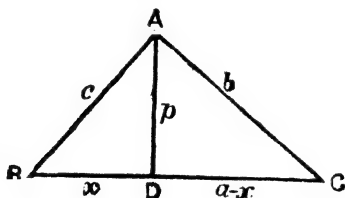
Note. A straight line is divided in **medial section** when the rectangle contained by the whole line and one of the parts is equal to the square on the other part.

7. If the sides BC, CA, AB of a triangle be respectively equal to a , b , and c units of length, and if

$$2s = a + b + c,$$

prove that the area of the triangle

$$= \sqrt{s(s-a)(s-b)(s-c)} \text{ units of area.}$$



Let the altitude AD of the triangle contain p units of length.

Let BD be $= x$ units of length; then CD $= (a - x)$ units of length.

From the $\triangle ADB$, we have $p^2 = c^2 - x^2$;

and from the $\triangle ADC$,

$$p^2 = b^2 - (a - x)^2.$$

$$\therefore c^2 - x^2 = b^2 - (a - x)^2.$$

$$\text{whence } x = \frac{a^2 - b^2 + c^2}{2a}.$$

$$\text{Hence, } p^2 = c^2 - x^2$$

$$= c^2 - \left(\frac{a^2 - b^2 + c^2}{2a} \right)^2$$

$$= \left\{ c + \frac{a^2 - b^2 + c^2}{2a} \right\} \left\{ c - \frac{a^2 - b^2 + c^2}{2a} \right\}$$

$$\begin{aligned}
 &= \frac{(c+a)^2 - b^2}{2a} \cdot \frac{b^2 - (c-a)^2}{2a} \\
 &= \frac{(c+a+b)(c+a-b)(b+c-a)(b-c+a)}{4a^2}.
 \end{aligned}$$

Now, since $2s = a + b + c$,

$$\begin{aligned}
 \therefore \quad c + a - b &= (a + b + c) - 2b = 2(s - b), \\
 b + c - a &= (a + b + c) - 2a = 2(s - a), \\
 a + b - c &= (a + b + c) - 2c = 2(s - c).
 \end{aligned}
 \quad \left. \vphantom{\begin{aligned} c + a - b \\ b + c - a \\ a + b - c \end{aligned}} \right\}$$

$$\therefore \quad p^2 = \frac{4s(s-a)(s-b)(s-c)}{a^2}$$

$$\therefore \quad p = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a}.$$

Hence, the number of units of area in the area of the $\triangle ABC$

$$= \frac{ap}{2} = \sqrt{s(s-a)(s-b)(s-c)}.$$

EXERCISE (18).

1. BCDE is a square on the hypotenuse BC of a right-angled triangle ABC. Prove that

$$AB^2 + AD^2 = AC^2 + AE^2.$$

2. O is a point within a triangle ABC, such that the three angles OAB, OBC, OCA, are equal to one another. Find the position of O.

3. If two triangles that are equal in area stand on the same base and on *opposite* sides of it, prove that the straight line joining their vertices is bisected by the base or the base produced.

Hence, prove that the medians of a triangle are concurrent.

4. E, F, G, H are the middle points of the sides AB, BC, CD, DA respectively of any quadrilateral ABCD. Shew that the quadrilateral EFGH is a parallelogram, and that its area is half that of ABCD.

* 5. If the acute angles of a right-angled triangle are 60° and 30° respectively, prove that the equilateral triangle described on the hypotenuse is equal to the sum of the equilateral triangles described on the other two sides.

6. If the angle C of a triangle ABC be 60° , prove that

$$AB^2 = BC^2 + CA^2 - BC \cdot CA.$$

7. If the angle C of a triangle ABC be 120° , prove that

$$AB^2 = BC^2 + CA^2 + BC \cdot CA.$$

8. In a triangle ABC the angles B and C are acute. If BE, CF be perpendiculars to AC, AB respectively, prove that

$$BC^2 = AB \cdot BF + AC \cdot CE.$$

9. Divide a given straight line into two parts so that the square on one of the parts may be double the square on the other.

10. Prove that in any quadrilateral the sum of the squares on the diagonals is equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides.

11. BC is the hypotenuse of a right-angled triangle ABC, and AD is perpendicular to BC. If b, c, h are the numerical measures of CA, AB, AD respectively, prove that

$$\frac{1}{h^2} = \frac{1}{b^2} + \frac{1}{c^2}.$$

12. Bisect a quadrilateral by a straight line drawn through a given point in one of its sides.

13. Taking any straight line as the unit of length, find a straight line of which the numerical measure is $\sqrt{6}$.

14. Calculate the area of a triangle the numerical measures of whose sides are 10, 17 and 21.

Hence find the numerical measure of the perpendicular on the greatest side from the opposite angular point.

15. The numerical measures of the parallel sides of a trapezium are 20 and 34, and those of the other two sides are 13 and 15. Calculate the area of the trapezium.

16. Describe a *rhombus* equal to a given rectilineal figure.

17. Construct a rectangle equal to the sum or difference of two given rectangles.

18. A and B are two fixed points. P and Q are two other points such that

$$PA^2 - PB^2 = QA^2 - QB^2.$$

Prove that the perpendiculars from P and Q upon AB are in the same straight line.

Hence, if A and B are two fixed points and a point P moves so that $PA^2 - PB^2$ is always the same, prove that the locus of P is a straight line perpendicular to AB.

19. The base BC of a triangle ABC is divided internally at D, so that

$$m.BD = n.CD; \text{ where } m \text{ and } n \text{ are integers.}$$

Prove that

$$m.AB^2 + n.AC^2 = m.BD^2 + n.DC^2 + (m+n).AD^2.$$

20. The base BC of a triangle ABC is divided *externally* at D, so that

$$m.BD = n.CD; \text{ where } m \text{ and } n \text{ are integers.}$$

Prove that

$$m.AB^2 - n.AC^2 = m.BD^2 - n.DC^2 + (m-n).AD^2.$$

21. If O be the centroid of a triangle ABC and P any other point, prove that

$$PA^2 + PB^2 + PC^2 = OA^2 + OB^2 + OC^2 + 3PO^2.$$

What point is that the sum of the squares of whose distances from the vertices of a given triangle is the *least*?

BOOK III

The Circle.

SECTION I.

FUNDAMENTAL IDEAS AND DEFINITIONS.

1. A circle is a plane figure bounded by one line which is such that all straight lines drawn to it from a certain point within the figure are equal to one another.

2. The bounding line of a circle is called its circumference.

3. The point within a circle from which all straight lines drawn to the circumference are equal is called the centre of the circle.

4. Any straight line drawn from the centre of a circle to the circumference is called a radius of the circle.

Note. 1. All radii of the same circle are equal to one another.

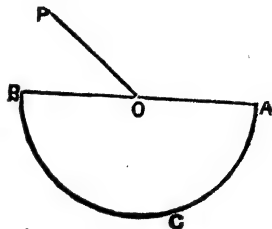
Note. 2. If a straight line be drawn from the centre of a circle equal to the radius, the extremity of the line will be on the circumference.

5. Any straight line drawn through the centre of a circle and terminated both ways by the circumference is called a diameter of the circle.

6. A **semi-circle** is the figure bounded by a diameter of a circle and the part of the circumference which it cuts off.

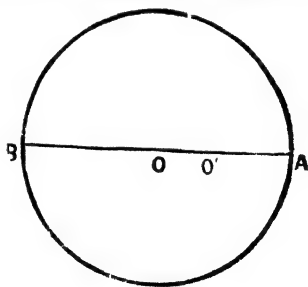
Note. 1. Every diameter divides a circle into two semi-circles.

Note. 2. Suppose the diameter AB of a semi-circle ACB to remain fixed, while the semi-circle is turned round AB until it falls on the side of AB remote from C . The distance of every point on the semi-circumference from O remains unaltered by this change of position. Hence, if OP be drawn in any direction on that side of AB which is remote from C and equal to a radius of the semi-circle, then P will be on the semi-circumference in its present position.



7. A circle cannot have more than one centre. It possible, let a circle have two centres,

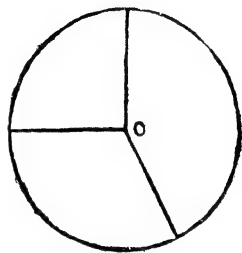
O and O' . Produce OO' both ways to meet the O^{ce} in A and B . Then, since $OA = OB$, we have $OA' = \frac{1}{2} AB$; and similarly, $O'A = \frac{1}{2} AB$. Therefore $OA = O'A$, which is impossible. Hence, if O is the centre of a circle, there is no other point satisfying the same condition.



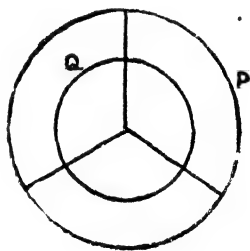
Note. If the circumference of two circles coincide their centres must also coincide. For, when the circumferences coincide, the two circles become one and the same circle. Hence, if the centres do not coincide, there will be two centres to one circle, which is impossible.

8. Two or more circles are said to be **concentric** when they have the same centre.

9. If two circles have equal radii, they are also equal in area. Suppose one of the circles to be taken up and so placed upon the other that the two circles become concentric, the point O being the common centre. Now, if a straight line be drawn from O in any direction and equal to the radius of one of the circles, it will also be equal to the radius of the other circle, and hence the extremity of the line will be on both the circumferences. Thus, the two circumferences will coincide *every where*, and will consequently enclose the same amount of surface.



10. If two circles have unequal radii, their areas also are unequal. Suppose one of the circles to be taken up and so placed upon the other that the two circles become concentric. Let P denote the circle which has the larger radius, and Q the other circle.



Then, whichever point on the O^c of Q may be taken, its distance from the common centre is less than the radius of P . Hence *every point on the O^c of Q , and consequently the entire circumference, lies within the circle P* . Clearly therefore the amount of surface enclosed by the O^c of Q is less than that enclosed by the O^c of P , which proves the proposition.

Note. Hence, *if two circles have equal areas their radii also are equal*. The radii *cannot be unequal*, for, in that case, the areas would be unequal too.

11. The word "circle" is often used to mean the "circumference of a circle". For instance, when we say that "one circle cannot cut another in more than two points," we mean that "the circumference of one circle cannot cut the circumference of another in more than two points."

12. Any part of the circumference of a circle is called an **ARC** of the circle.

13. The finite straight line joining any two points on the circumference of a circle is called a **CHORD** of the circle.

Note 1. A diameter is also a chord.

Note. 2. Every chord, which is not a diameter, divides the circumference into two *unequal* arcs, of which the greater one is called the **major arc**, and the lesser, the **minor arc**.

Note 3. Two arcs of a circle are said to be **conjugate** to each other when they together make up the whole circumference. Hence two *conjugate* arcs stand on opposite sides of the same chord.

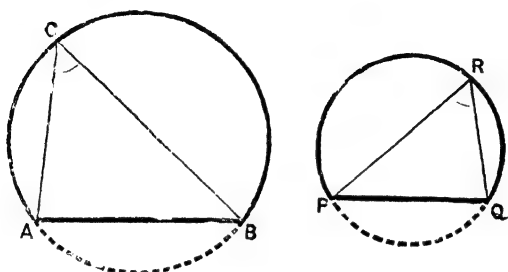
14. A **segment** of a circle is the figure bounded by an arc and the chord joining the extremities of the arc.

Note. 1. A semi-circle is also a segment.

Note 2. The chord of a segment is also called the *base* of the segment.

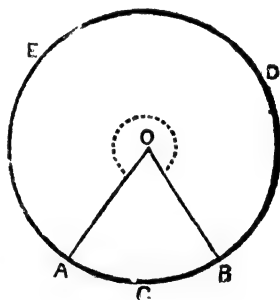
15. An **angle in a segment** is the angle contained by two straight lines drawn from any point in the arc of the segment to the extremities of its chord.

16. **Similar segments** are those which contain equal angles. Thus, in the following diagram, the segments ACB and PRQ are *similar*, because the \angle s ACB and PRQ are equal.



17. A **sector** of a circle is the figure bounded by an arc and the two radii drawn to the extremities of the arc.

Thus, in the adjoining diagram, the figure $AOBC$ is a sector of the $\odot ABD$. The figure $AOBDE$ is also a sector.



Note 1. The angle between the two bounding radii of a sector is called the **angle of the sector**. Hence the angle of the sector $AOBDE$ is the re-entrant angle AOB .

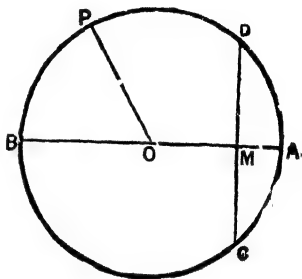
Note 2. A sector becomes a semi-circle when the angle of the sector becomes equal to two right angles.

18. A figure is said to be **symmetrical about a line** if the part of the figure on one side of the line coincides completely with the part on the other side when the figure is folded about that line.

Note 1. It is easy to see that an isosceles triangle is symmetrical about the bisector of its vertical angle.

Note 2. The line about which a figure is symmetrical is called a **line of symmetry** in the figure. In other words, if a straight line and a figure be so related that, when the figure is folded about the line, the part of the figure on one side of it coincides completely with the part on the other side, then the line is called a *line of symmetry in the figure*.

19. A circle is symmetrical about any diameter. Let O be the centre of the circle ABC of which AB is a diameter, as in the adjoining diagram. Let the circle be folded about AB so that the semi-circumference ACB falls on the side of AB remote from C . Now, if from O a straight line OP be drawn in any direction to meet the semi-circumference ADB in P , then P will also be on the semi-circumference ACB in its present position, because OP is a radius of the circle.



Thus *any* straight line drawn through O to meet the semi-circumference ADB will meet both the semi-circumferences in the same point. Hence the two semi-circumferences will coincide *everywhere*.

Note 1. Two points on the circumference of a circle are said to be *symmetrically opposite* with regard to a diameter, if they coincide with one another when the circle is folded about that diameter. For instance, if the points C and D , in the above diagram, coincide when the circle is folded about AB , then C and D are *symmetrically opposite points* with regard to AB . Each of the points C and D is also said to be the *image* of the other with regard to the diameter BA .

III.] FUNDAMENTAL IDEAS AND DEFINITIONS. 169

Note 2. *The chord joining two symmetrically opposite points with regard to a diameter is bisected at right angles by that diameter.*

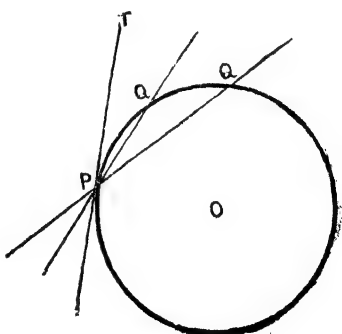
In the preceding diagram, let CD cut AB at M . Then, when the circle is folded about AB , and C coincides with D , it is evident that MC coincides with MD and is therefore equal to it; clearly also the angle OMC coincides with the angle OMD and is therefore equal to it. Hence AB bisects CD and is also perpendicular to it.

20. Any unlimited straight line that cuts the circumference of a circle is called a **secant** of a circle.

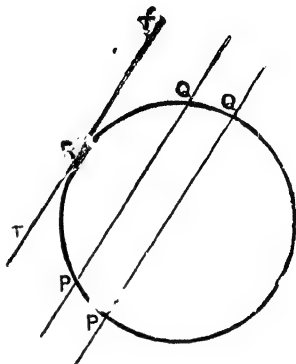
Note. Hence, a chord when produced both ways becomes a *secant*, or, in other words, a chord is that portion of a secant which lies within the circle.

21. If a secant of a circle moves so that the two points in which it cuts the circumference continually approach one another, then in the position which it ultimately takes up when the two points coincide, the secant becomes what is called a **tangent** to the circle; and it is said to *touch* or be the *tangent* to, the circle at the point at which those two points coincide.

Note 1. In the joining diagram, let PQ be a secant of the circle whose centre is O . Let P remain fixed while Q moves along the circumference towards P ; then the secant PQ will continually change its position. If PT be the position which the secant assumes when Q coincides with P , then PT is the *tangent* to the circle at the point P



Note 2. Let P, Q be any two points on the circumference of a circle. Suppose that the secant PQ moves in such a manner that the points P and Q approach nearer and nearer to some point R on the arc PQ , and ultimately coincide at R . If TRT' be the position of the secant when P and Q coincide, then TRT' is the *tangent* to the circle at the point R .



Note 3. From the above it is clear that when a straight line becomes a *tangent* to a circle at any point, it does not cut the circumference at that point. Hence a tangent to a circle is an *unlimited straight line which meets the circumference but does not cut it at the point of meeting*.

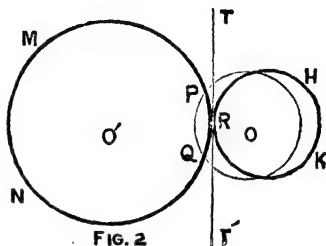
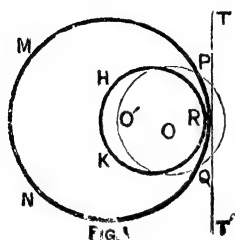
Note 4. The point at which the tangent to a circle meets it, is called the *point of contact* of the tangent.

Note 5. A tangent may also be regarded as a secant cutting the circumference in two *coincident* points. In other words, a tangent is a secant of which the chord-portion does not exist.

22. If the circumference of one circle passes through two near points on the circumference of another, and if one of the circles moves, while the other remains fixed, so that the two points come nearer and nearer to each other, then in its ultimate position, when the two points coincide, the moving circle is said to *touch* the other at the point of coincidence of those two points.

Note 1. In the diagram on the next page, let O and O' be the centres of the two circles which cut each other in the points P and Q . If the first circle moves, while the second remains fixed, so that the points P and Q come nearer and nearer to one another, and if

HRK be the ultimate position of the moving circle when **P** and **Q** coincide at **R**, then the circle **HRK** is said to *touch* the circle **RMN** at the point **R**.



Note 2. Hence it is clear that two circles may be said to *touch* each other when they *meet* but *do not cut* each other at the point of meeting.

Note 3. It may also be said that one circle touches another when the circumference of the former passes through two *coincident* points on the circumference of the latter.

Note 4. In the above diagram, let **TRT'** be the ultimate position of the secant **PQ**, when **P** and **Q** coincide at **R**.

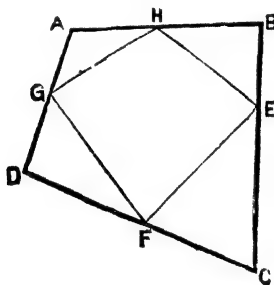
Then **TRT'** is clearly a tangent to each of the two circles, because it passes through two *coincident* points on each circumference. Hence, *when two circles touch each other, they have a common tangent at the point of contact.*

Note 5. Two circles are said to touch each other **internally**, or to have **inteanal contact** when the centres of the circles are on the *same* side of the point of contact, as in fig. (1); and they are said to touch each other **externally**, or to have **external contact**, when the centres are on opposite sides of the point of contact, as in fig. (2).

23. When all the angular points of a rectilineal figure lie on the circumference of a circle, the rectilineal figure is said to be **inscribed in the circle**, and the circle is said to be **circumscribed about the rectilineal figure**.

24. When each side of a rectilineal figure is a tangent to a circle, the rectilineal figure is said to be **circumscribed about the circle**, and the circle is said to be **inscribed in the rectilineal figure**.

25. If each vertex of one rectilineal figure lies on a side of another and there is one vertex of the first figure on each side of the second, then the first figure is said to be **inscribed in** the second then the second is said to be **circumscribed about** the first. Thus, in the following diagram, the quadrilateral HEFG is *inscribed in* the quadrilateral ABCD, and the quadrilateral ABCD is *circumscribed about* the quadrilateral HEFG.



EXERCISE (19).

1. Let O be the centre of a circle whose radius is two inches. If from O any straight line OP be drawn equal to two inches in length, prove that the point P is neither inside the circle nor outside it.

2. If A and B be two concentric circles such that the radius of A is smaller than that of B , prove that every point on the circumference of A is within the circle B .

3. If A and B be two concentric circles such that the radius of A is equal to that of B, prove that every point on the circumference of A is on the circumference of B.

4. Deduce, from examples 2 and 3, that if two circles cut each other they *can not have the same centre*.

5. AB is a chord of a circle of which the centre is O. If P be any point in AB and Q any point in AB produced, prove that OP is less than, and OQ is greater than, the radius of the circle.

6. When does a segment become a semi-circle ?

7. When does a sector become a segment ?

8. Every chord divides a circle into two segments ; when are the segments equal ?

9. Divide a circle into two equal sectors.

10. Prove that a semi-circle is symmetrical about a straight line drawn through the centre at right angles to the diameter.

11. Prove that a sector is symmetrical about the bisector of its angle.

12. If a chord be perpendicular to a diameter, prove that the extremities of the chord are symmetrically opposite with regard to that diameter.

13. When a secant becomes a tangent, what becomes of the extremities of the chord-portion of the secant ? Where then is the middle point of the chord-portion ?

14. Two circles that touch each other may also be regarded as cutting each other in two points. How? Prove that two circles that touch each other have a common tangent at the point of contact.

15. When is one rectilinear figure said to be inscribed in another? Show that if one or other of the two conditions were omitted from the definition, the inner figure could not necessarily be regarded as having the same number of sides as the outer figure.

16. AB is a diameter of a circle of which the centre is O ; and P is a point on the diameter between O and B . If C is any other point on the circumference, prove that PC is less than PA but greater than PB .

If D be a point on the arc AC , prove that PD is greater than PC .

17. AB is a diameter of a circle, and P is a point on BA produced. If C is any other point on the circumference, prove that PC is greater than PA but less than PB .

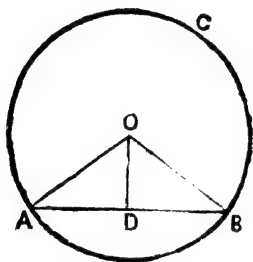
SECTION II.

THEOREMS.

Theorem 1. (Euc. III. 3.)

A straight line drawn from the centre of a circle to bisect a chord which is not a diameter, is at right angles to the chord ;

Conversely, the perpendicular to a chord from the centre bisects the chord.



Let O be the centre, and AB a chord, of the $\odot ABC$.

(i) Let D be the mid.-point of AB .

To prove that OD is \perp to AB .

Join OA, OB .

Proof. In the \triangle s OAD, OBD , we have

(1) $AD = DB$, (Hyp.)

(2) OD common,

and (3) $OA = OB$; (Radii of the \odot)

\therefore the two \triangle s are congruent. (Bk. I, Th. 7.)

Hence the $\angle ODA = \angle ODB$;

and $\therefore OD$ is \perp to AB .

Q. E. D.

(ii) Let OD be \perp to AB .

To prove that $AD = DB$.

Join OA, OB .

Proof. In the *right-angled* \triangle s ODA, ODB , we have

(1) OD common,

and (2) the hypotenuse $OA =$ the hypotenuse OB ;

(Radii)

\therefore the two \triangle s are congruent.

Hence $AD = DB$.

Q. E. D.

Cor. 1. *A straight line cannot cut a circle in more than two points.*

For, in the above figure, if the chord AB were to meet the circle again at some other point E , then D would be the middle point of AB as well as of AE , which is impossible.

Cor. 2. *The middle points of any system of parallel chords of a circle lie in a straight line passing through the centre.*

For, if a str. line be drawn from the centre \perp to one of the chords, it will be \perp to all chords, and will \therefore intersect all the chords at their middle points.

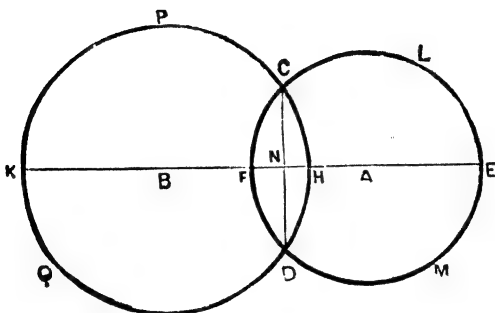
Note. 1. Of the many bisectors of AB there is *only* one which is *also* perpendicular to it ; and it is on *this* bisector, and on *no other*, that O must lie, because O is equi-distant from A and B . (Th. 25, Bk. I.). Hence, if it is known that OD is a bisector of AB , we at once conclude that OD is also perpendicular to it, because O can not lie on any other bisector.

Note 2. Of the many perpendiculars to **AB** there is *only* one which also bisects **AB**; and it is on *this* perpendicular, and on *no other*, that **O** must lie, because **O** is equidistant from **A** and **B** (Th. 25, Bk. I). Hence, if it is known that **OD** is perpendicular to **AB**, we at once conclude that **OD** also bisects it, because **O** can not lie on any other perpendicular.

Note 3. If a perpendicular be drawn to **AB** through its middle point, it *must pass through* **O**; because, no point that is equidistant from **A** and **B** can lie outside this perpendicular.

Theorem 2.

If two circles cut each other in two points, the line joining their centres bisects the common chord at right angles.



Let the \odot s whose centres are A and B cut each other in the points C and D.

Join AB. Also join CD ; then CD is the common chord of the two \odot s.

To prove that AB bisects CD at right angles.

Let AB be produced both ways ; and let it cut the \odot A in E and F, and the \odot B in H and K.

Let N be the point in which AB and CD intersect.

Proof. Suppose the figure to be folded about the line EK, so that the upper portion of the figure falls on the side of EK remote from C.

Then, since the \odot A is symmetrical about the diameter EF, the semi- \odot^{ce} FLE will entirely coincide with the

semi- \bigcirc^{ce} **FME** ; and, for a similar reason, the semi- \bigcirc^{ce} **HPK** will entirely coincide with the semi- \bigcirc^{ce} **HQK**.

Hence, the point **C** will fall upon the semi- \bigcirc^{ce} **FME** and also upon the semi- \bigcirc^{ce} **HQK** ; it will therefore coincide with the pt. **D**.

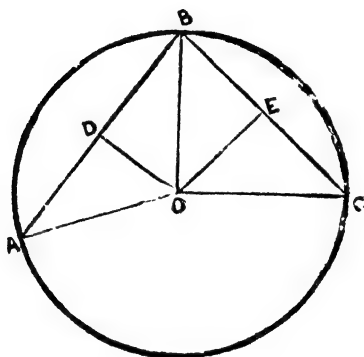
Hence **NC = ND**, and the $\angle \text{BNC} = \text{the } \angle \text{BND}$; which shews that **AB** bisects **CD** at right angles. Q. E. D.

Note 1. The proposition may be also proved as follows : Through **N**, the mid-pt. of **CD**, draw the str. line **KNE** perpendicular to **CD**. Then, since the pt. **A** is equidistant from **C** and **D**, it *must lie* on the str. line **KE** (Th. 25, Bk. I.) ; and, for a similar reason, the pt. **B** also must lie on **KE**. Hence the str. line **AB** coincides with the str. line **KE**, and \therefore it bisects **CD** at right angles.

Note 2. It may be also deduced as a corollary to Theorem 1 : Let **N** be the mid-pt. of **CD** ; join **AN** and **BN**. Then **AN** and **BN** are both perpendicular to **CD** (Th. 1) ; the \angle s **ANC**, **BNC** are therefore together = two rt. angles, and hence **AN** and **NB** are in the same str. line. Hence, the str. line **AB** bisects **CD** and is also perpendicular to it.

Theorem 3.

There is one circle, and one only, which passes through three given points not in a straight line.



Let A, B, C be three given points which do not lie in one str. line.

To prove (i) that a circle can be drawn through A, B, C; and (ii) that there is only one circle passing through them.

Proof. (i) Let DO and EO, which are the \perp bisectors of AB and BC, meet at O. Join OA, OB, OC.

Then, since O is on the \perp bisector of AB, we have $OA = OB$;

and, for a similar reason, $OB = OC$.

$$\therefore OA = OB = OC.$$

Hence the \odot described with centre O and radius OA will pass through all the three pts. A, B, C.

(ii) If a \odot be described with centre O and a radius different from OA , it will pass through *none* of the pts. A, B, C .

If a \odot be described with centre O' , *different from* O , it will *not* pass through *all* the three pts. A, B, C ; for, a point other than O can *not* be equidistant from those three pts.

Hence the *only* \odot that can pass through the three pts. A, B , and C is the \odot described with centre O and radius OA .

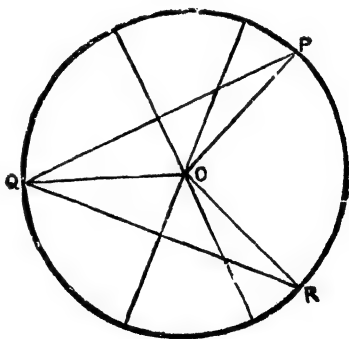
Q. E. D.

Cor. 1. One circle can not cut another in more than two points. For, if one \odot has three points in common with another, then there will be *two* \odot s. passing through the same three points, which is impossible

Cor. 2. If from a point O within a circle three equal straight lines OP, OQ, OR can be drawn to the circumference, then O must be the centre of the circle.

Since the pt. O is equidistant from the pts. P, Q, R , it must be the pt. of intersection of the \perp bisectors of PQ and QR ; and there is *no other* pt. besides this

which can be equidistant from those three pts. Hence the centre of the given \odot , which certainly is equidistant from the pts. P, Q, R , on the circumference, must coincide with O .



Note 1. A circle is said to be **described** (or **circumscribed**) **about a triangle** when it passes through the vertices of the triangle. The circle described about a triangle is called the **circum-circle** of the triangle ; and the centre and radius of this circle are respectively called the **circum-centre** and the **circum-radius** of the triangle.

Note 2. The *circum-centre* of a triangle is evidently the point of intersection of the perpendicular bisectors of any two of its sides ; and the *circum-radius* is the distance of any vertex of the triangle from the circum-centre.

EXERCISE (20).

1. AB is a chord of a circle of which the centre is O . Prove that the straight line drawn from O to the middle point of AB passes through the middle point of any other chord that is parallel to AB .

2. Prove that in a circle the centre and the middle points of any number of chords that are parallel to one another lie in one straight line.

3. If two chords of a circle bisect each other, prove that their point of intersection must be the centre of the circle.

Hence, prove that the diagonals of a parallelogram inscribed in a circle are equal and that the parallelogram is a rectangle.

4. If two circles, whose centres are A and B , have one point P in common, prove that they have also another point Q in common, where Q is such that PQ is bisected at right angles by AB .

5. If any number of circles pass through two given points, prove that their centres all lie in one straight line.

6. If the chords AB , AC of a circle are equally inclined to the radius to A , prove that $AB = AC$.

7. Prove that the circum-centre of a right-angled triangle is the mid-point of its hypotenuse.

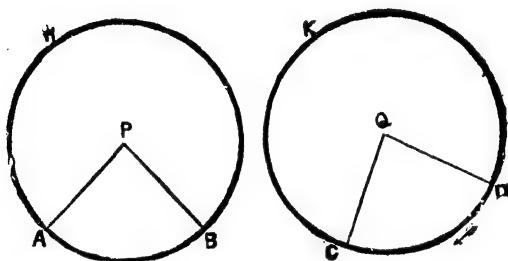
8. If two circles have two points in common, prove that any two parallel straight lines drawn through the common points and terminated by the circumferences are equal.

9. Construct a triangle whose sides are 2, 3 and 4 inches respectively, and find the point which is equidistant from the angular points of the triangle.

Theorem 4. (EUC. III. 26 AND 27.)

In two equal circles, if there be two arcs, one in each, subtending equal angles at the centres, then these two arcs are equal.

Conversely, if there be two equal arcs, one in each, then the angles which they subtend at the centres are equal.



Let ABH and CDK be two equal \odot s whose centres are P and Q ; and let the arcs AB and CD subtend the \angle s APB and CQD respectively at the centres.

(i) If the $\angle APB$ be equal to the $\angle CQD$, to prove that the arc AB is equal to the arc CD .

Apply the $\odot CDK$ to the $\odot ABH$, turning the former in the direction of the arrow-head, so that the centre Q may coincide with the centre P , and QC may fall upon PA .

Then, since the $\angle CQD =$ the $\angle APB$, QD will fall upon PB .

Hence, since $QC = PA$ and $QD = PB$, the pts. C and D must coincide with the pts. A and B respectively.

Also, since the two \odot s have now become concentric and they have equal radii, the two \odot^{ces} must coincide *everywhere*.

Hence the arc CD entirely coincides with the arc AB and is \therefore equal to it. Q. E. D.

(ii) If the arc AB be equal to the arc CD , to prove that the $\angle APB =$ the $\angle CQD$.

Apply the $\odot CDK$ to the $\odot ABH$, turning the former in the direction of the arrow-head, so that the centre Q may coincide with the centre P , and QC may fall upon PA .

Then, since $QC = PA$, the pt. C must coincide with the pt. A ;

Also, since the \odot s have now become concentric and they have equal radii, the two \odot^{ces} coincide *everywhere*.

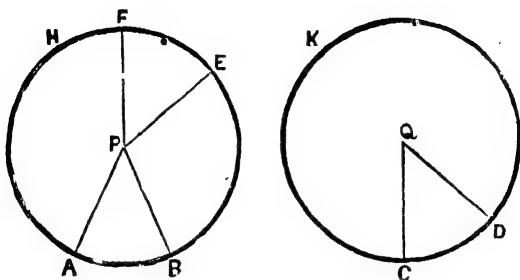
Hence, since the arc $CD =$ the arc AB , the pt. D must coincide with the pt. B

Now, Q coinciding with P and D with B , QD coincides with PB ; and consequently the $\angle CQD$ coincides with the $\angle APB$.

Hence, the $\angle CQD =$ the $\angle APB$.

Q. E. D.

Cor. 1. If in the same circle ABH , the arcs AB and EF subtend equal \angle s APB , EPF at the centre, then the arc $AB =$ the arc EF .



Let CDK be a \odot whose centre is Q and radius = PA, and let the \angle CQD contained by the radii QC and QD be = the \angle APB (or the \angle EPF). Clearly then the arc AB = the arc EF, because each of them = the arc CD.

Cor. 2. *If in the same circle ABH, the arc AB be = the arc EF, then the \angle s APB, EPF, which these arcs subtend at the centre, are equal.*

Let CDK be a \odot whose centre is Q and radius = PA, and let the arc CD be = the arc AB (or the arc EF). Clearly then the \angle APB = the \angle EPF, because each of them = the \angle CQD.

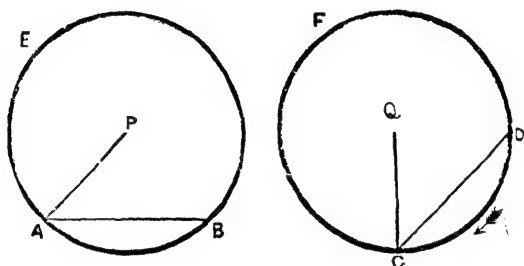
Cor. 3. *In equal circles, sectors which have equal angles are equal; also, sectors which have equal arcs are equal.*

If the \odot CDK be applied to the \odot ABH so that the \angle CQD coincides with the \angle APB, then the arc CD also coincides with the arc AB. Hence the sector QCD entirely coincides with the sector PAB and is \therefore equal to it. Similarly, if the arc CD be = the arc AB, it follows that the sector QCD = the sector PAB.

Theorem 5. (EUC. III. 29 AND 28.)

In two equal circles, if there be two equal arcs, one in each, then the chords of these two arcs are equal ;

Conversely, if there be two equal chords, one in each, then the arcs which they cut off are equal, the major arc being equal to the major arc and the minor to the minor.



Let $\odot ABE$ and $\odot CDF$ be two equal \odot s of which the centres are P and Q .

(i) Let the arc AB be = the arc CD .

Join AB, CD .

To prove that the chord AB is equal to the chord CD .

Proof. Join PA, QC .

Apply the $\odot CDF$ to the $\odot ABE$, turning the former in the direction of the arrow-head, so that the centre Q may coincide with the centre P and QC may fall upon PA .

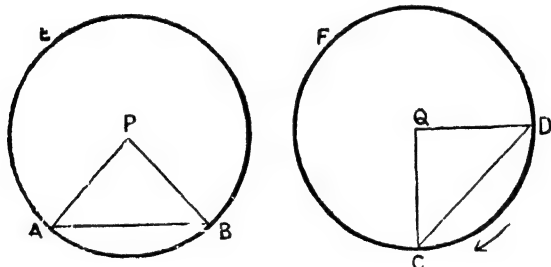
Then, since $QC = PA$, the pt. C must coincide with the pt. A .

Since the \odot s have now become concentric and they have equal radii, the two \odot es coincide *everywhere*. Hence the arc CD falls upon the arc AB; and consequently, since the arcs are equal, the pt. D coincides with the pt. B.

Hence the chord CD coincides with the chord AB and is \therefore equal to it. Q. E. D.

(ii) Let the chord AB be = the chord CD.

To prove that the minor arc AB is equal to the minor arc CD, and the major arc AEB is equal to the major arc CFD.



Proof. Join PA, PB, QC, QD.

Since PA, PB, AB are respectively = QC, QD, CD, therefore the two \triangle s PAB and QCD are congruent.

Hence the \angle APB = the \angle CQD.

Now, apply the \odot CDF to the \odot ABE, turning the former in the direction of the arrow-head, so that the centre Q may coincide with the centre P, and QC may fall upon PA.

Then, since the \angle CQD = the \angle APB, QD will fall upon PB.

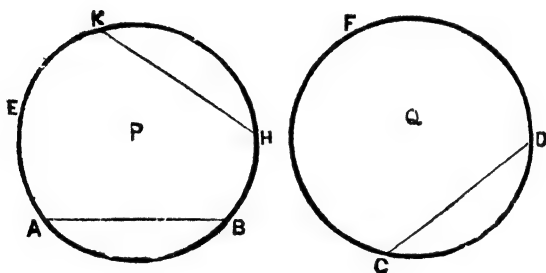
Hence, since $QC = PA$ and $QD = PB$, the pts. C and D must coincide with the pts. A and B respectively.

Also, since the two \odot s have now become concentric and they have equal radii, the two \odot 's coincide *everywhere*.

Hence the arc CD entirely coincides with the arc AB and is \therefore equal to it.

Also the arc CFD entirely coincides with the arc AEB is \therefore equal to it. Q. E. D.

Cor. 1. *If in the same circle ABE, the arc OB be equal to the arc HK, then the chord AB is equal to the chord HK.*



Let CDF be an equal \odot having its centre at Q, and let the arc CD be = the arc AB (or the arc HK). Clearly then the chord AB = the chord HK, because each of them = the chord CD.

Cor. 2. *If in the same circle ABE, the chord AB be equal to the chord HK, then the minor arc AB is equal to the minor arc HK, and the major arc AEB is equal to the major arc HEK.*

Let CDF be an equal \odot having its centre at Q, and let the chord CD be = the chord AB (or the chord HK).

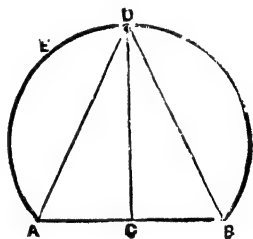
Clearly then the minor arc AB = the minor arc HK , because each of them = the minor arc CD . Also the major arc AEB = the major arc HEK , because each of them = the major arc CFD .

Cor. 3. *In equal circles, segments which have equal arcs are equal ; also, segments which have equal chords and which are both greater than or both less than a semi-circle, are equal.*

For, if the \odot CDF be so applied to the \odot ABE that the arc CD coincides with the arc AB , then the segment CD entirely coincides with the segment AB and is \therefore equal to it. Again, if the chord CD be = the chord AB , the \odot CDF may be so applied to the \odot ABE that the arc CD coincides with the arc AB and the arc CFD with the arc AEB . Hence the segment CD entirely coincides with the segment AB and is \therefore equal to it ; also the segment CFD entirely coincides with the segment AEB and is \therefore equal to it.

Cor. 4. *If C be the middle point of the chord AB of an arc AEB , and if CD be drawn perpendicular to AB meeting the arc in D , then D is the middle point of the arc AEB .*

Since DC produced is a diameter of the \odot of which

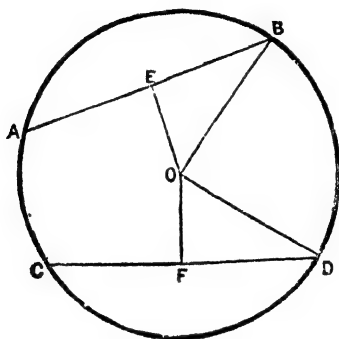


AEB is an arc, the arcs DA and DB are clearly both less than the semi- \odot of the \odot . Now, if DA and DB be joined, the two \triangle s ACD and BCD are evidently congruent ; whence $DA = DB$, and \therefore the arc DA = the arc DB .

Theorem 6. (Euc. III. 14.)

Equal chords of a circle are equidistant from the centre.

Conversely, chords which are equidistant from the centre are equal.



Let AB and CD be two chords of a \odot of which the centre is O .

Let OE , OF be the \perp s from O upon AB and CD ; then OE , OF are respectively the *distances* of AB and CD from O .

(i) If $AB = CD$, to prove that $OE = OF$.

Proof. Join OB , OD .

Since OE is \perp to AB , $\therefore AE = EB$; (Th. 1.)

and $\therefore EB = \frac{1}{2} AB$.

Similarly, $FD = \frac{1}{2} CD$.

Hence, $EB = FD$.

Now, in the rt. angled \triangle s OEB , OFD , we have

(1) the hypotenuse $OB =$ the hypotenuse OD ,
and (2) $EB = FD$;

\therefore the two Δ s are congruent.

Hence, $OE = OF$.

Q. E. D.

(ii) If $OE = OF$ to prove that $AB = CD$.

Proof. Join OB, OD .

In the rt. angled Δ s OEB, OFD , we have

(1) the hypotenuse $OB =$ the hypotenuse OD ,

and (2) $OE = OF$;

\therefore the two Δ s are congruent.

Hence, $EB = FD$. (a)

Now, OE being \perp to AB , E is the mid-pt. of AB ;

$\therefore AB = 2EB$.

Similarly, $CD = 2FD$.

Hence, from (a), $AB = CD$.

Q. E. D.

Cor. The middle points of equal chords of a circle lie on the circumference of a concentric circle.

For, the distances of the middle points of the chords from the centre of the \odot are respectively the \perp s upon the chords from the centre, and are \therefore equal to one another.

Note. It is easy to see that $OE^2 + EB^2 = OF^2 + FD^2$. Hence, if $OE > OF$, we have $EB < FD$ and $\therefore AB < CD$; also, if $OE < OF$, we have $EB > FD$ and $\therefore AB > CD$. Thus, of any two chords in a circle, that which is nearer to the centre is the greater; and conversely, that which is greater is nearer to the centre than the other. It is also clear, that $OB > EB$. Hence the diameter through B and any diameter is $> AB$; and similarly, the diameter through D and any diameter is $> CD$. Thus of all chords in a circle the greatest is any that passes through the centre.

EXERCISE (21).

1. A, B, C are three points on the circumference of a circle of which the centre is O. If the angle AOC be three times the angle AOB, prove that the arc AC is also three times the arc AB.

2. P and Q are two points on the circumference of a circle of which the centre is O. If the $\angle POQ = 120^\circ$, prove that the arc PQ is one-third of the whole circumference, and that the chord PQ is a side of an equilateral triangle inscribed in the circle.

Hence construct an equilateral triangle with its vertices on the circumference of a circle whose radius is 2 inches.

3. If AB and CD are two diameters of a circle, perpendicular to each other, prove that the quadrilateral ACBD is both equilateral and equiangular.

4. If a regular hexagon be inscribed in a circle, prove that each side of the figure is equal to a radius of the circle.

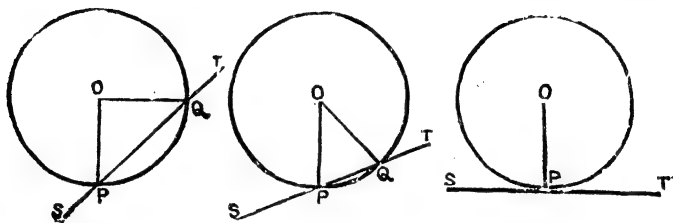
5. If two equal circles intersect so that the centre of one circle is on the circumference of the other, prove that one-third of each circumference lies within the other.

6. AB, AC, AD are three chords of a circle, taken in order, AB being a diameter. Prove that AB is greater than AC and AC is greater than AD. Prove also that the chord AD is more distant from the centre than AC.

7. O is the centre of a circle of which the radius is 2 inches, and A, B, C are three points each at a distance of one inch from O. Construct the chords that are bisected respectively at A, B, C and prove that these chords are equal to one another. Also verify this fact by actual measurement.

Theorem 7. (Euc. III. 18.)

The tangent at any point of a circle is perpendicular to the radius through the point.



Let P be any point on the \odot^{ce} of a \odot whose centre is O ; join OP .

To prove that the tangent at P is perpendicular to OP .

Proof. Let ST be a secant through P , cutting the \odot again at Q .

Join OQ .

Since $OQ = OP$, \therefore the $\angle OPQ =$ the $\angle OQP$.

Now, the $\angle OPS$ is the supplement of the $\angle OPQ$,
and the $\angle OQT$ is the supplement of the $\angle OQP$. }

Hence the $\angle OPS = \angle OQT$.

Let the secant ST be turned about the pt. P so that the pt. Q moves along the arc QP and gradually approaches P .

Then, in every position of the secant the $\angle OPS =$ the $\angle OQT$.

Ultimately, when Q coincides with P , the secant ST becomes the tangent at P ; and the $\angle OQT$ becomes the $\angle OPT$.

Hence, when ST is the tangent at P , the $\angle OPS$
 $\angle OPT$;

and these are adjacent angles ;

$\therefore OP$ is \perp to ST .

That is, the tangent at P is \perp to OP . Q. E. D.

Cor. 1. *There cannot be more than one tangent to a circle at any given point on its circumference.*

For, if there were two tangents to a \odot at any given pt. P on its \bigcirc , each of them would be \perp to OP , and thus there would be two \perp s to OP at P , which is impossible.

Cor. 2. *If through any point, on a circle a straight line be drawn perpendicular to the radius to that point, it must be the tangent to the circle at that point.*

If PT' be drawn \perp to OP , PT' must coincide in direction with PT ; for, there would otherwise be two perpendiculars to OP at P , which is impossible.

Cor. 3. *If from the point of contact of a tangent to a circle a straight line be drawn perpendicular to the tangent, it must pass through the centre of the circle.*

If PO' be drawn \perp to PT , PO' must coincide in direction with PO ; for, there would otherwise be two perpendiculars to PT at P , which is impossible.

Cor. 4. *If through the centre of a circle a straight line be drawn perpendicular to a tangent to the circle, it must meet the tangent at the point of contact.*

If through O a straight line be drawn \perp to the tangent SPT , it must coincide in direction with OP ; for, there would otherwise be two perpendiculars to SPT through O , which is impossible.

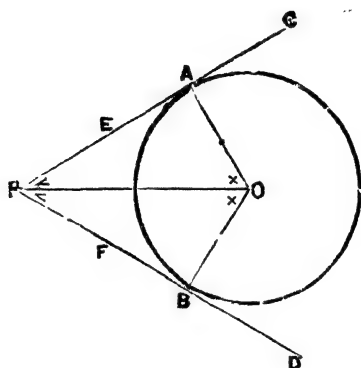
Cor. 5. *Every point on a tangent to a circle except the point of contact, is outside the circle.*

Let SPT be a tangent to a \odot whose centre is O , the pt. of contact being P . Then, since OP is \perp to ST , every other point on ST , besides P , is at a greater distance from O than OP , *i.e.*, at a greater distance from O than the radius of the \odot ; which proves the corollary.

Note. The proposition may also be proved as follows : The straight line drawn from the centre of a circle to the middle point of the chord-portion of a secant is always perpendicular to the secant (Th. 1). When the secant becomes a tangent, the chord-portion becomes reduced to the mere point of contact, and so the middle point of the chord-portion too becomes identical with the point of contact. Hence the straight line drawn from the centre to the point of contact of a tangent (which may be regarded as the straight line drawn from the centre to the middle point of the chord-portion of the tangent) is perpendicular to the tangent.

Theorem 8.

If two tangents to a circle meet at a point, the portions of the tangents intercepted between the point and the circle are equal, they subtend equal angles at the centre of the circle, and they are also equally inclined to the straight line drawn from the point to the centre of the circle.



Let EAC and FBD be two tangents to a \odot whose centre is O, the points of contact being A and B respectively. Let the tangents meet at the point P.

Join OA, OB, OP.

To prove that $PA = PB$.

the $\angle AOP = \text{the } \angle BOP$.

and the $\angle APO = \text{the } \angle BPO$.

Proof. Since OA and OB are the radii drawn to the pts. of contact, the tangents PAC and PBD are respectively \perp to OA and OB. (Th. 7.)

Hence the \angle s OAP, OBP are rt. \angle s,

Now, in the rt.-angled \triangle s OAP, OBP, we have

(1) The hypotenuse OP common,

and (2) $OA = OB$;

\therefore The two triangles are congruent.

Hence, $PA = PB$,	}	Q. E. D.
the $\angle AOP =$ the $\angle BOP$,		
and the $\angle APO =$ the $\angle BPO$.		

Note 1. When a tangent is drawn to a circle from a point outside it, *the portion of the tangent intercepted between the point and the circle* is often spoken of as *the tangent drawn from the point to the circle*. Hence, Theorem 8 may also be enunciated as follows: "The two tangents drawn to a circle from an external point are equal and subtend equal angles at the centre of the circle, and they are also &c."

Note 2. If **AB** is joined, the line **AB** is called the **chord of contact** of the tangents from P to the circle.

Theorem 9. (EUC. III. 11 AND 12.)

If two circles touch the point of contact lies on the straight line through the centres.

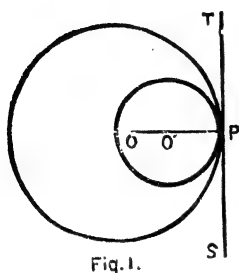
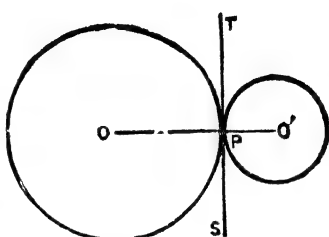


Fig. 1.



• Fig. 2.

Let the \odot s whose centres are O and O' touch at the pt. P .

To prove that O , O' and P are in the same str. line.

Proof. The two \odot s have a common tangent at P .
(Sec I., Art. 22, Note 4.)

Let TPS be the common tangent ; join OP , $O'P$.

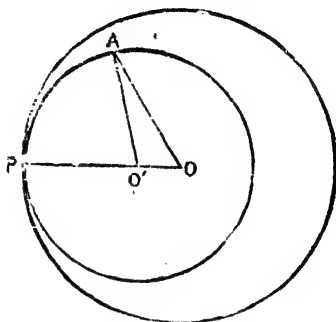
Then OP , $O'P$ being radii drawn to the pt. of contact, each of the \angle s TPO , TPO' is a rt. \angle . (Th. 7.)

Hence in fig. (1), O' is on the line joining PO ;

and in fig. (2), O' is on OP produced. Thus, in both the cases, the three pts. O , O' , P are in one str. line. Q. E. D.

Cor. 1. *If two circles touch each other internally, every point on the circle of smaller radius, except the point of contact lies inside the other circle.*

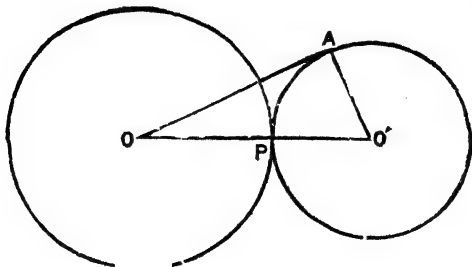
Let the \odot s whose centres are O and O' touch each other internally at P , the second \odot being of smaller radius. Then O, O', P are in one str. line and $OP > O'P$. Take *any* pt. A , other than the pt. P , on the \odot of smaller radius : join $OA, O'A$. Now, OA is



less than the sum of OO' and $O'A$, and is \therefore less than OP . Thus the distance of A from the centre of the first \odot is less than its radius and $\therefore A$ is *inside* the first \odot ; which proves the corollary.

Cor. 2. *If two circles touch each other externally, every point on either circle, except the point of contact, lies outside the other.*

Let the \odot s whose centres are O and O' touch each other externally at P . Then O, O', P are in one str. line. Take *any* pt. A , other than the pt. P , on the second \odot ; join $OA, O'A$. Now, $OA + O'A > OO'$ but $O'A = O'P$; $\therefore OA > OP$. Thus the distance of A from the centre of the first \odot is greater than its radius, and $\therefore A$ is *outside* the first \odot ; which proves that *every* pt. on the second \odot ,



except P , is outside the first. Similarly, every pt. on the first \odot ; except P , is outside the second.

Note 1. The main proposition may also be proved as follows: If two circles cut each other in two points, their centres and the mid-pt. of the common chord lie in one str. line: (Th. 2.). When the circles touch, the two pts. of intersection coincide; the common chord, therefore, becomes reduced to the mere pt. of contact, and consequently the mid-pt. of the chord also becomes identical with the point of contact. Hence, when the circles touch, their centres and the point of contact must lie in one str. line.

Note 2. *If two circles have a common point on the line joining their centres, the circles touch each other at that point.*

For, if the circles intersect in two distinct points, the line joining the centres passes through the mid-pt. of the common chord and never through one or other of the pts. of intersection.

Note 3. *If two circles have the same tangent at a common point they touch each other at that point.*

For, if the circles intersect in two distinct pts., they cannot have common tangent at either of those pts., they can only have a common *secant* passing through the two pts. of intersection.

EXERCISE (22).

1. If two circles are concentric, prove that any chord of the outer circle which touches the inner is bisected at the point of contact.

2. If any number of equal circles touch a given straight line on the same side of it, prove that their centres all lie in one straight line.

3. If P be a point from which tangents are drawn to a circle whose centre is O , prove that OP is the perpendicular bisector of the chord of contact.

4. If any number of circles pass through the same point and touch one another at that point, prove that their centres all lie in one str. line.

5. If any number of circles touch each of two intersecting straight lines, prove that their centres all lie in one straight line.

6. If a parallelogram be described about a circle, prove that the sum of one pair of its opposite sides is equal to that of the other pair. Hence prove that any two adjacent sides of the parallelogram are equal.

7. If a quadrilateral be such that the sum of one pair of its opposite sides is equal to that of the other pair, prove that a circle which touches any three sides of this quadrilateral must also touch the fourth.

8. If three equal circles touch one another at P, Q, R , prove that the tangents at P, Q, R meet at a point.

Theorem 10. (Euc. III. 20.)

The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.

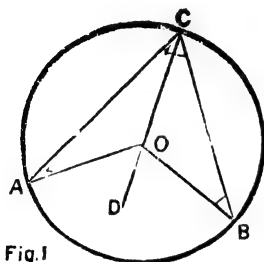


Fig.1

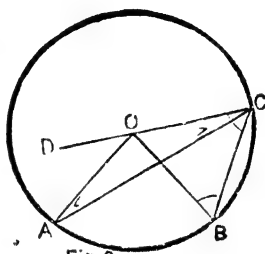


Fig.2

Let O be the centre of a \odot of which AB is an arc, and C is a pt. on the remaining part of the \odot^{ce} .

Join OA, OB, CA, CB .

To prove that the $\angle AOB =$ twice the $\angle ACB$.

Proof. Join CO and produce it to any pt D .

Since $OB = OC$, \therefore the $\angle OCB =$ the $\angle OBC$.

Hence, the $\angle DOB =$ the $\angle OCB +$ the $\angle OBC$
 $=$ twice the $\angle OCB$.

Similarly, the $\angle AOD =$ twice the $\angle OCA$.

Hence, in fig. (1),

the $\angle DOB +$ the $\angle AOD =$ twice the $\angle OCB +$
twice the $\angle OCA$,

i.e., the $\angle AOB =$ twice the $\angle ACB$,

and in fig. (2),

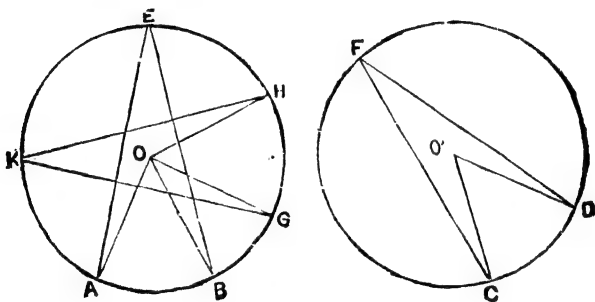
the $\angle DOB$ minus the $\angle AOD =$ twice the $\angle OCB$
 minus twice the $\angle OCA$,

i.e., the $\angle AOB =$ twice the $\angle ACB$.

Q. E. D.

Note. If the arc AB be equal to the semi-circumference of the circle, the $\angle AOB$ becomes a *straight-angle*; and if the arc AB be greater than the semi-circumference, then the $\angle AOB$ becomes a *reflex angle*. The same proof will hold for these two cases as for fig. (1).

Cor. 1. In equal circles (or in the same circle), arcs which subtend equal angles at the circumferences (or circumference) are equal.



Let O, O' be the centres of two equal \odot s. If the arcs AB and CD subtend equal \angle s AEB, CFD at the \odot^{cs} , then they also subtend equal \angle s at the centres, (because the \angle s $AOB, CO'D$ are respectively doubles of the \angle s AEB, CFD); and \therefore the arc $AB =$ the arc CD . (*Th. 4.*)

If in the same \odot , the arcs AB and GH subtend equal \angle s AEB, GKH at the \odot^{c} , then they also subtend equal \angle s at the centre, (for the \angle s AOB, GOH are respectively doubles of the \angle s AEB, GKH); and \therefore the arc $AB =$ the arc GH . (*Th. 4. Cor. 1.*)

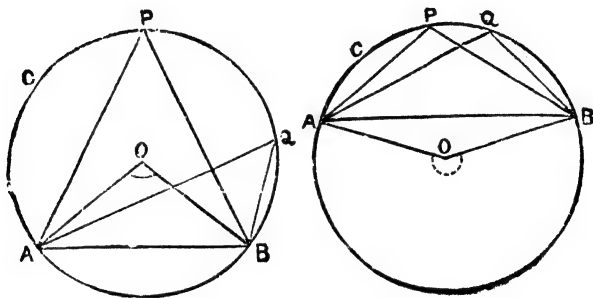
Cor. 2. In equal circles (or in the same circle), angles at the circumferences (or circumference) which stand on equal arcs are equal.

In the preceding diagram, if the arc AB be = the arc CD , then the $\angle AOB$ = the $\angle CO'D$, (*Th. 4.*); and \therefore the $\angle AEB$ = the $\angle CFD$, because these \angle s are respectively halves of the \angle s AOB , $CO'D$.

Again, if the arc AB be = the arc GH , then the $\angle AOB$ = the $\angle GOH$, (*Th. 4, Cor. 2.*); and \therefore the $\angle AEB$ = the $\angle GKH$, because these \angle s are respectively halves of the \angle s AOB , GOH .

Theorem 11. (Euc. III. 21.)

Angles in the same segment of a circle are equal.



Let $\angle APB$ be any \angle in the segment ACB of the \odot whose centre is O . Let $\angle AQB$ be *any other* \angle in the same segment.

To prove that the $\angle APB =$ the $\angle AQB$.

Proof. Join OA, OB .

The arc AB of the \odot subtends the $\angle AOB$ at the centre, and the $\angle APB$ at the pt. P on the remaining part of the \odot^{ce} ;

\therefore the $\angle APB =$ half the $\angle AOB$. (Th. 10.)

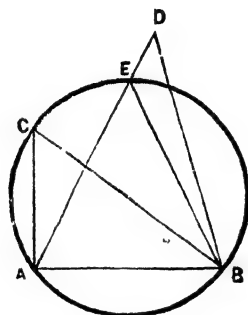
Similarly, the $\angle AQB =$ half the $\angle AOB$.

Hence, the $\angle APB =$ the $\angle AQB$. Q. E. D.

Note. In the second circle of the preceding figure, the segment ACB being less than a semi-circle, the angle AOB is a *reflex* angle.

Theorem 12.

If the line joining two points subtends equal angles at two other points on the same side of it, the four points lie on a circle.



Let C and D be two pts. on the same side of the str. line AB, and let the $\angle ACB$ be = the $\angle ADB$.

To prove that the four points A, B, C, D lie on the circumference of a circle.

Proof. A \odot will pass through *any three* of the four pts. (Th. 3.)

Hence, all that is required is to prove that the \odot passing through three of the pts. will also pass through the remaining pt.

If the \odot passing through A, B, C do not also pass through D, then it will cut AD or AD produced at some pt. E.

Join EB.

Now, the \angle s ACB and AEB being in the same segment,

the $\angle ACB = \text{the } \angle AEB.$

But the $\angle ACB = \text{the } \angle ADB;$ (Hyp.)

\therefore the $\angle AEB = \text{the } \angle ADB,$ which is impossible.

Hence the \odot passing through A, B, C must also pass through D. Q. E. D.

Cor. *Triangles standing on the same base, and on the same side of it, with equal vertical angles, have their vertices on the circumference of a circle of which the given base is a chord.*

Let AB be the common base, and C, D, E, F, &c., the vertices of the \triangle s; then, since the $\angle ACB = \text{the } \angle ADB,$ the \odot passing through A, B, C also passes through D. Similarly, the \odot passing through A, B, C also passes through E; and so on. Thus the pts. D, E, F, &c., all lie on the \odot that passes through the pts. A, B, C; which proves the corollary.

Note. The fore-going corollary and Theorem 11 are converse propositions. For, the corollary may be stated in a slightly altered form, thus:—

If triangles standing on the same base and on the same side of it, have equal vertical angles, then their vertices lie on the arc of a segment of which the given base is the chord; whilst Theorem 11 may also be enunciated as follows:—

If triangles standing on the same base, and on the same side of it, have their vertices on the arc of a segment of which the given base is the chord, their vertical angles are equal.

EXERCISE (23).

1. $ABCD$ is a circle of which the centre is O . If the sum of the angles ADB , BDC be equal to one right angle, prove that the points A , O , C lie in one straight line.

2. If in a circle two arcs be such that they subtend complementary angles at the circumference, prove that the two arcs are together equal to half the circumference of the circle.

3. If the chords AB , CD of a circle intersect at right angles, prove that the sum of the arcs AC , BD is equal to half the circumference of the circle.

4. P is any point within a circle of which the centre is O . If AB , CD be any two chords passing through P , prove that the angle APC is equal to half the sum of the angles AOC , BOD .

5. AB , CD are two chords of a circle such that when produced they meet at P . If O is the centre of the circle, prove that the angle APC is equal to half the difference between the angles AOC , BOC .

6. AB is the common chord of two equal circles. If any straight line drawn through B meet the circumferences in P and Q , prove that the triangle PAQ is isosceles.

7. Two given circles intersect in the points A and B . If two straight lines pass through B of which one meets the circumferences in P and Q and the other in M and N , prove that PQ and MN subtend equal angles at A .

8. ABC is a triangle and D , E , F are the middle points of the sides opposite to A , B , C . If M be the foot

of the perpendicular from A upon BC , prove that each of the angles FDE , FME is equal to the angle BAC ; and hence prove that a circle will pass through the four points D , E , F , M .

9. $ABCD$ is a circle. The bisectors of the angles CAB , CBA meet at P , and the bisectors of the angles DAB , DBA meet at Q . Prove that the four points A , Q , P , B lie on the circumference of the same circle.

Theorem 13. (EUC. III. 31.)

The angle in a semi-circle is a right angle; the angle in a segment greater than a semi-circle is less than a right angle; and the angle in a segment less than a semi-circle is greater than a right angle.

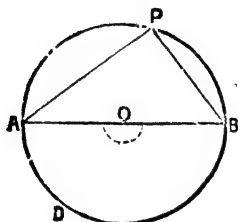


Fig 1

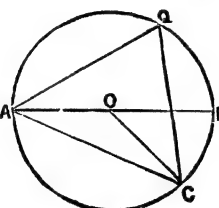


Fig 2

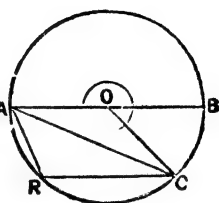


Fig 3

Let AB be a diameter of a \odot of which the centre is O .

(i) Let APB be any \angle in the semi- \odot APB , as in fig. (i).

To prove that the $\angle APB$ is a right angle.

Proof. The semi- \odot^{ce} ADB subtends the *straight angle* AOB at the centre, and the $\angle APB$ at the pt. P on the remaining part of the \odot^{ce} ;

$$\begin{aligned} \therefore \text{ the } \angle APB &= \text{half the str. } \angle AOB && (\text{Th. 10.}) \\ &= \text{half of two rt. } \angle \text{s.} \\ &= \text{one right angle.} && \text{Q. E. D.} \end{aligned}$$

(ii) Let AC be a chord of the \odot , as in fig. (2).

Then the segment ABC is $>$ a semi- \odot ; let AQC be any \angle in this segment.

To prove that the $\angle AQC$ is less than a right angle.

Proof. Join OC .

The $\angle AQC = \text{half the } \angle AOC$. (Th. 10.)

But the $\angle AOC$ is $<$ two rt. \angle s ;

\therefore the $\angle AQC$ is $<$ one right angle. Q. E. D.

(iii) Let AC be a chord of the \odot , as in fig. (3).

Then the segment ARC is $<$ a semi- \odot ;

let ARC be any \angle in this segment.

To prove that the $\angle ARC$ is greater than a rt. angle.

Proof. Join OC .

The arc ABC subtends the *reflex* $\angle AOC$ at the centre, and $\angle ARC$ at the pt. R on the remaining part of the \odot^{ce} ,

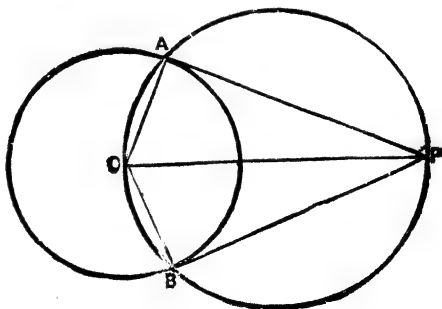
\therefore the $\angle ARC = \text{half the reflex } \angle AOC$. (Th. 10.)

But the *reflex* $\angle AOC$ is $>$ two rt. \angle s ;

\therefore the $\angle ARC$ is $>$ one right angle. Q. E. D.

Note. From the first part of this proposition we deduce the construction of a tangent to a circle from any point outside it.

Let P be a point outside the \odot whose centre is O . Join OP , and let the \odot on OP as diameter cut the given circle at A and B ; join OA , OB , AP , BP . Then the $\angle OAP$ being in a semi- \odot , is a right angle. Thus AP is \perp to the radius OA



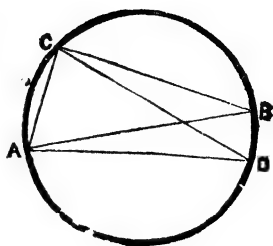
and $\therefore AP$ is a tangent to the given \odot .

(Th. 7, Cor. 2.)

Similarly BP is a tangent to the given \odot .

Cor. 1. *If a chord AB of a circle subtends a right angle at the point C on the circumference, then AB is a diameter of the circle.*

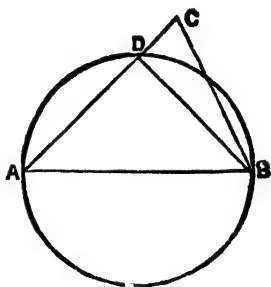
If AB be not a diameter of the \odot , let a different chord AD be the diameter through the pt. A, as in the accompanying diagram.



Join CD. Then the $\angle ACD$, being in a semi- \odot , is a rt. \angle , and \therefore equal to the $\angle ACB$, which is impossible. Hence AD is not a diameter; and similarly, no other chord through A, different from AB, can be a diameter. Hence AB is the diameter through A.

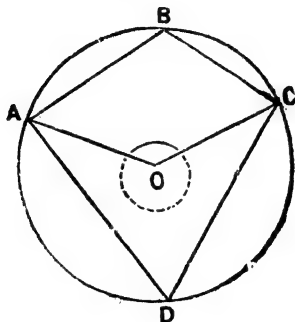
Cor.. 2. *If a straight line AB subtends a right angle at any point C, then the circle described on AB as diameter passes through C.*

If the \odot does not pass through C, let it cut AC or AC produced at some pt. D, as in the accompanying diagram. Join DB. Then the $\angle ADB$, being in semi- \odot , is a rt. \angle , and \therefore equal to the $\angle ACB$, which is impossible. Hence the \odot must pass through C.



Theorem 14. (EUC. III. 22.)

The opposite angles of any quadrilateral inscribed in a circle are supplementary.



Let $ABCD$ be a quadrilateral inscribed in a \odot whose centre is O .

To prove that (i) the $\angle ABC +$ the $\angle ADC =$ two right angles.

and (ii) the $\angle BAD +$ the $\angle BCD =$ two right angles.

Proof. Join OA, OC .

The arc ABC subtends the $\angle AOC$ at the centre, and the $\angle ADC$ at the pt. D on the remaining part of the \odot^{ce} ;

\therefore the $\angle ADC =$ half the $\angle AOC$. (Th. 10.)

Again, the arc ADC subtends the *reflex* $\angle AOC$ at the centre, and the $\angle ABC$ at the pt. B on the remaining part of the \odot^{ce} .

\therefore the $\angle ABC =$ half the *reflex* $\angle AOC$. (Th. 10.)

Hence, the $\angle ADC +$ the $\angle ABC =$ half the sum of the $\angle AOC$ and the *reflex* $\angle AOC$.

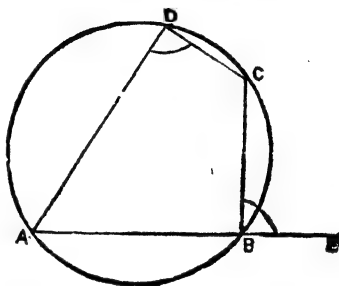
= half of four right angles

= two right angles.

Similarly, the $\angle BAD +$ the $\angle BCD =$ two right angles.

Cor. 1. *If the side AB of a quadrilateral ABCD inscribed in a circle be produced to E, the angle EBC is equal to the angle ADC.*

The $\angle EBC$ is supplementary to the $\angle ABC$; also the $\angle ADC$ is supplementary to the $\angle ABC$. Hence, the $\angle EBC =$ the $\angle ADC$.



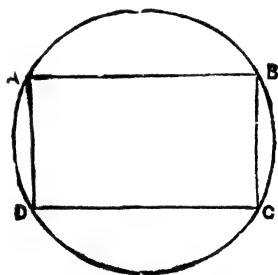
Cor. 2. *If a circle passes through the angular points of a parallelogram, the parallelogram must be a rectangle.*

Let a \odot pass through the angular pts. of the par^m . ABCD.

The $\angle A$ is = the $\angle C$

(*opp. \angle s of a par^m .*);

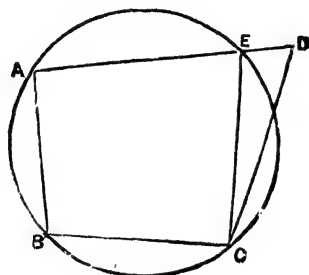
also, the sum of the \angle s A and C is two rt. \angle s, because the quadl. ABCD is inscribed in a \odot . Hence, each of the \angle s A and C is a rt. \angle , which proves the corollary.



Note. A quadrilateral is said to be **cyclic** when a circle can pass through its angular points. Four or more points are said to be **concyclic** when a circle can pass through them all. Three or more points are said to be **collinear** when they lie in one straight line.

Theorem 15.

If two opposite angles of a quadrilateral are supplementary, a circle can be described through its angular points.



Let the \angle s B and D of the quadr. ABCD be together equal to two rt. \angle s.

To prove that the four points A, B, C, D are concyclic.

Proof. A \odot will pass through the pts. A, B, C. If this \odot does not pass through D, let it cut AD or AD produced at some pt. E. Join EC.

Now, the quadr. ABCE being cyclic, the \angle AEC is the supplement of the \angle ABC ;

and the \angle ADC is also the supplement of the \angle ABC. (Hyp.)

\therefore the \angle AEC = the \angle ADC, which is impossible.

Hence the \odot which passes through A, B, C must also pass through D. Q. E. D.

Note. It may be observed that if the \angle s B and D are supplementary the \angle s A and C are also supplementary, because the four angles of a quadrilateral are together equal to four right angles.

Cor. *If, when the side AB of a quadrilateral ABCD is produced to E the angle EBC be equal to the angle ADC. then the four points A, B, C, D are concyclic.*

EXERCISE (24).

1. Two circles intersect at A and B. If AP and AQ are diameters, prove that the three points P, B, Q are collinear.

2. ABC is a triangle, and the circle described on AB as diameter cuts BC at D. Prove that the circle described on AC as diameter also passes through D.

3. Circles are described on the sides of a quadrilateral as diameters. Prove that the common chord of two of these circles which are adjacent is parallel to the common chord of the other two.

4. If a triangle is inscribed in a circle, prove that the sum of the angles in the three segments exterior to the triangle is equal to four right angles.

5. If a quadrilateral is inscribed in a circle, prove that the sum of the angles in the four segments exterior to the quadrilateral is equal to six right angles.

6. ABC is a triangle, and D is the middle point of BC. If BP, CQ be drawn perpendicular to AC, AB respectively, prove that the perpendicular bisector of PQ will pass through D. Prove also that the triangles ABC and APQ are equiangular.

7. In the figure of the preceding example prove that the angle BCQ is equal to the angle BPQ.

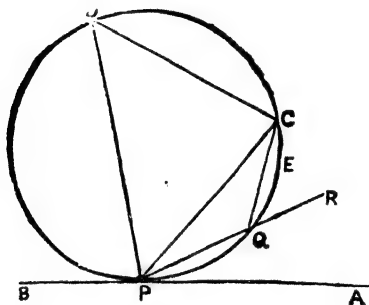
8. Prove that the straight lines, which bisect any angle of a quadrilateral inscribed in a circle and the opposite exterior angle, meet on the circumference of the circle.

9. A quadrilateral is inscribed in a circle. If the bisectors of two of its opposite angles meet the circle in P and Q, prove that PQ is a diameter.

10. Prove that the four straight lines, bisecting the angles of any quadrilateral, form a cyclic quadrilateral,

Theorem 16. (Euc. III. 32.)

If a straight line touch a circle, and from the point of contact a chord be drawn, the angles which this chord makes with the tangent are respectively equal to the angles in the alternate segments.



Let the straight line AB touch the \odot PCD at the point P; and let PC be a chord drawn through P, dividing the \odot into the segments CDP and CEP.

To prove that the $\angle CPA$ = the angle in the segment CDP, and the $\angle CPB$ = the angle in the segment CEP.

Proof. In the arc CEP take a point Q close to P; join PQ and produce it to R. Join CQ, DC, DP.

Now, the quadl. PQCD being cyclic, the $\angle CQR$ = the $\angle CPD$; (Th. 14, Cor. 1.)

and this equality holds good however close to P the point Q may be taken.

Suppose the secant PR to be turned about P, so that Q moves towards P; then, ultimately, when Q coincides with P, the $\angle CQR$ becomes the $\angle CPA$.

Hence the $\angle CPA = \text{the } \angle CDP$.

By similar construction and reasoning, the $\angle CPB = \text{the angle in the segment CEP}$. Q. E. D.

Alternative Proof.

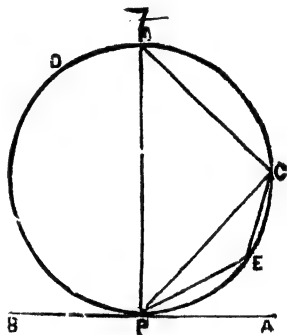
Draw the diameter PF through P. Join FC, CE, EP.

The $\angle PCF$, being in a semi-
 \odot , is a rt. \angle :

Hence, the $\angle CFP + \text{the } \angle FPC = \text{one rt. } \angle$;

also, since FP passes through the centre. \therefore the $\angle FPA$ is a rt. angle. (Th. 7.)

\therefore the $\angle FPA = \text{the } \angle CFP + \text{the } \angle FPC$.



Hence, taking away the common $\angle FPC$, the $\angle CPA = \text{the } \angle CFP$.

Again, the quadl. PECF being cyclic, the $\angle CEP$ is the supplement of the $\angle CFP$;

also, the $\angle CPB$ is the supplement of the $\angle CPA$.

But, supplements of equal \angle s are equal;

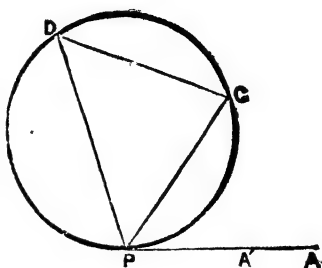
\therefore the $\angle CPB = \text{the } \angle CEP$.

Q. E. D.

Cor. If PDC is a triangle, and if through P a straight line PA be drawn on the side of PC remote from D, such that the angle CPA is equal to the angle CDP, then PA is the tangent at P to the circum-circle of the triangle PDC.

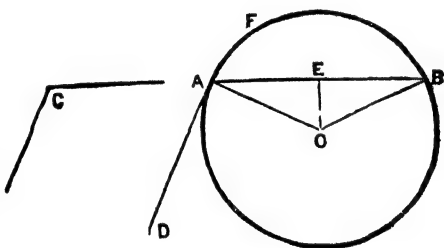
If PA' be the tangent to the circum-circle at P . then the $\angle CPA' =$ the $\angle CDP$; but the $\angle CDP =$ the $\angle CPA$. (Hyp.)

\therefore the $\angle CPA' =$ the $\angle CPA$; hence PA' and PA coincide, i.e., PA is the tangent at P to the circle FDC .



Note. We can apply this proposition to construct on a given straight line AB a segment of a circle containing an angle equal to a given angle C .

Make the $\angle BAD =$ the $\angle C$. Let the perpendicular bisector of AB and the perpendicular to AD through A meet at O . Join OB . Since O is on the \perp bisector of AB , $\therefore OA = OB$.



Hence the \odot described with O as centre and OA as radius will pass through B .

Now, since AD is \perp to the radius OA .

$\therefore AD$ touches the \odot at A .

(Th. 7, Cor. 2.)

Hence the \angle in the segment $AFB =$ the $\angle BAD =$ the $\angle C$. Thus, AFB is the required segment.

Theorem 17. (EUC. III. 35 AND 36.)

If two chords of a circle intersect either inside or outside the circle, the rectangle contained by the parts of the one is equal to the rectangle contained by the parts of the other.

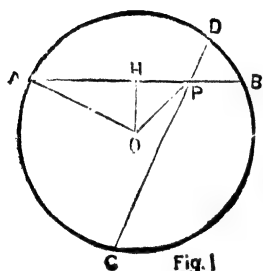


Fig. 1

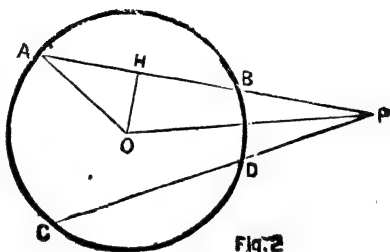


Fig. 2

Let the chords AB, CD of the \odot whose centre is O intersect inside or outside the \odot at P.

To prove that $AP \cdot PB = CP \cdot PD$.

Proof. Let OH be the \perp from O upon AB.

Join OP, OA.

Since OH is \perp to AB, $\therefore AH = HB$. (Th. 1.)

(i) When P is inside the \odot , as in fig. (1).

$$\begin{aligned} OA^2 - OP^2 &= (AH^2 + HO^2) - (PH^2 + HO^2) \\ &= AH^2 - PH^2 \\ &= (AH + PH)(AH - PH) \text{ (Th. 12, Bk. II.)} \\ &= (AH + PH)(BH - PH) \\ &= AP \cdot PB; \end{aligned}$$

and similarly, $OC^2 - OP^2 = CP \cdot PD$.

Hence, since $OA^2 = OC^2$, we have $AP \cdot PB = CP \cdot PD$.

(ii) When P is outside the \odot , as in fig. (2),

$$\begin{aligned}
 OP^2 - OA^2 &= (PH^2 + HO^2) - (AH^2 + HO^2) \\
 &= PH^2 - AH^2 \\
 &= (PH + AH)(PH - AH) \quad (\text{Th. 12, Bk. II.}) \\
 &= (PH + AH)(PH - HB) \\
 &= AP.PB;
 \end{aligned}$$

and similarly, $OP^2 - OC^2 = CP.PD$.

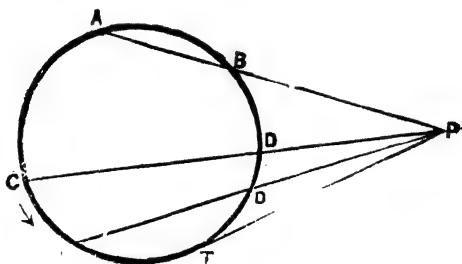
Hence, since $OA^2 = OC^2$, we have

$$AP.PB = CP.PD,$$

Q. E. D.

Cor. 1. If from any point P without a circle two straight lines be drawn, one of which cuts the circle in B and A , and the other touches it at T , then, $PA.PB = PT^2$.

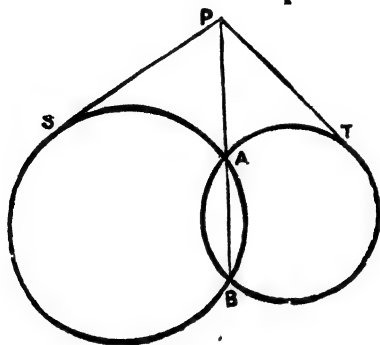
Let the secant PDC cut the \odot in D and C . Suppose PC to be turned about P in the direction of the arrow-head, so that the pts. C and D continually approach each other.



In every position of the secant PC , we have $PA.PB = PC.PD$; hence, ultimately, when the pts. C and D coincide at T and the secant PDC becomes the tangent PT , we must have $PA.PB = PT^2$.

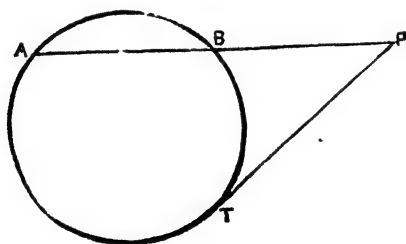
Note. Hence it follows that if two circles cut each other in two

points, the tangents drawn to the circles from any point on the common chord produced are equal.



The above figure explains itself. We have $PS^2 = PT^2$, because each of them $= PA \cdot PB$; $\therefore PS = PT$.

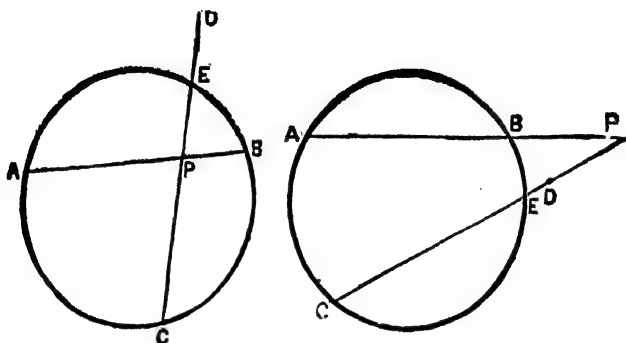
Cor. 2. If from any point P without a circle two straight lines be drawn, one cutting the circle in B and A and the other meeting it at T , so that $PA \cdot PB = PT^2$, then PT touches the circle at T .



If PT do not touch the \odot at T , let PT produced cut the \odot again in some other pt. S . Then, $PT \cdot PS = PA \cdot PB$, and $\therefore PT \cdot PS = PT^2$, which is impossible.

Hence PT touches the \odot at T .

Cor. 3. *If two straight lines AB and CD, or both of them produced, intersect at P, so that $AP \cdot PB = CP \cdot PD$, then the four points A, B, C, D are concyclic.*



A \odot will pass through the pts. A, B, C. If this \odot do not pass through D, let it cut CD or CD produced at E. Then we must have $AP \cdot PB = CP \cdot PE$, and $\therefore CP \cdot PD = CP \cdot PE$, which is impossible. Hence the \odot that passes through A, B, C must also pass through D.

EXERCISE (25).

1. P is the middle point of an arc APB of a circle. Prove that the tangent at P is parallel to the chord AB.

2. If an equilateral triangle be inscribed in a circle, prove that the tangents at its angular points will form another equilateral triangle.

3. Two circles intersect at A and B; and through P, any point on the circumference of one of them, str. lines PAC, PBD are drawn to cut the other \odot at C and D. Prove that CD is \parallel to the tangent at P.

4. ABC is a triangle right-angled at C; and from C a perpendicular CD is drawn to the hypotenuse. Prove that the square on CD is equal to the rectangle AD. DB.

5. Construct a triangle, having given the base, the vertical angle, and the length of the str. line drawn from the vertex to the middle pt. of the base.

6. If two circles intersect, prove that their common chord produced bisects their common tangent.

7. Two \odot s touch each other externally. Prove that any str. line drawn through the pt. of contact cuts off similar segments from the two \odot s.

8. ABC is a triangle, and DE is drawn parallel to BC cutting the sides in D and E. Prove that the circles circumscribed about the triangles ADE and ABC have a common tangent at A.

9. ABC is a triangle; and BE, CF are the perpendiculars from B and C upon the opposite sides, intersecting each other at O. Prove that (i) $BO.OE = CO.OF$, (ii) $AB.AF = AC.AE$.

10. AB is the common chord of two given circles; and through any point in AB two chords are drawn, one in each circle. Prove that the four extremities of these chords all lie on the circumference of a circle.

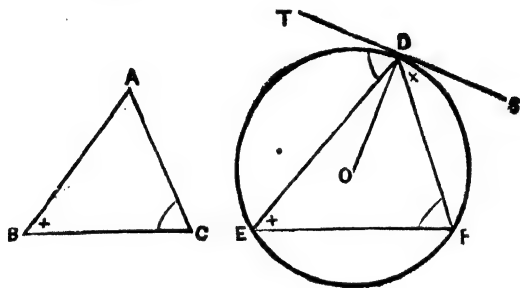
11. ABC is triangle right-angled at C , and CD is the perpendicular from C upon the hypotenuse. Prove that $AB \cdot AD = AC^2$.

SECTION III.

PROBLEMS.

Problem 1. (Euc. IV. 2.)

In a given circle to inscribe a triangle equiangular to a given triangle.



Let O be the centre of the given \odot , and ABC the given \triangle .

It is required to inscribe in the circle (\odot) a triangle equiangular to the triangle ABC .

CONS. Take any radius OD , and through D draw $TS \perp$ to OD ; then TS is the tangent at D .

Draw the chord DE making the $\angle TDE = \text{the } \angle ACB$;
and draw the chord DF making the $\angle SDF = \text{the } \angle ABC$.

Join EF .

Then DEF is the reqd. \triangle .

Proof. Since TS touches the \odot at D ,

\therefore the $\angle TDE = \text{the } \angle DFE$, in the alt. segment;
and the $\angle SDF = \text{the } \angle DEF$, in the alt. segment.

Hence the $\angle DFE = \text{the } \angle ACB,$ }
 and the $\angle DEF = \text{the } \angle ABC ;$ }

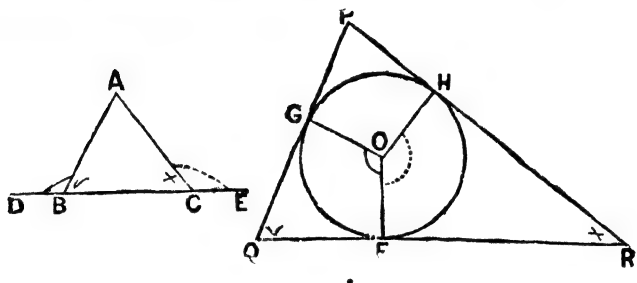
\therefore the remaining $\angle EDF = \text{the remaining } \angle BAC.$

Thus, the $\triangle DEF$ inscribed in the given \odot is equiangular to the $\triangle ABC.$ Q. E. F.

Note To inscribe an equilateral triangle in a given circle, all that we have to do is to construct an equilateral triangle outside the circle and then inscribe in the circle a triangle equiangular to the triangle constructed, in the manner indicated above. In this case therefore, each of the angles TDE and SDF would be equal to an angle of an equilateral triangle, i.e. equal to one-third of two right angles.

Problem 2. (EUC. IV. 3.)

About a given circle to circumscribe a triangle equiangular to a given triangle.



Let O be the centre of the given \odot , and ABC the given \triangle .

It is required to circumscribe about the circle (O) a triangle equiangular to the triangle ABC .

Cons. Produce BC both ways to D and E .

Take any radius OF of the \odot .

Draw the radius OG making the $\angle FOG = \text{the } \angle ABD$; and draw the radius OH making the $\angle FOH = \text{the } \angle ACE$.

Draw tangents to the \odot at the pts. F, G, H , forming the $\triangle PQR$.

Then PQR is the reqd. \triangle .

Proof. The \angle s OFQ and OGQ being rt. \angle s, the remaining \angle s FOG and GQF of the quadl. $OFQG$ are supplementary.

Now, the $\angle ABC$ is the supplement of the $\angle ABD$, and the $\angle GQF$ is the supplement of the $\angle FOG$;

but the $\angle ABD =$ the $\angle FOG$,

\therefore the $\angle ABC =$ the $\angle PQR$. }

Similarly, the $\angle ACB =$ the $\angle PRQ$. }

\therefore the remaining $\angle QPR =$ the remaining $\angle BAC$.

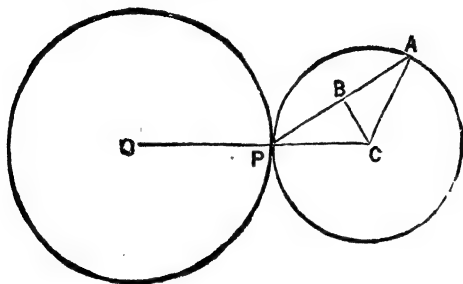
Thus, the $\triangle PQR$ circumscribed about the given \odot is equiangular to the $\triangle ABC$. Q. E. F.

Note 1. To circumscribe an equilateral triangle about a given circle, all that we have to do is to construct an equilateral triangle outside the circle and then circumscribe about the circle a triangle equiangular to the triangle constructed, in the manner indicated above. In this case, therefore, each of the angles **FOG**, **FOH** would be equal to the supplement of an angle of an equilateral triangle, *i.e.*, equal to two-thirds of two right angles.

Note 2. An equilateral triangle is evidently a *regular* figure of three sides; for an equilateral \triangle is also equiangular. Hence, from Note to Problem 1 and Note 1 above, we know how to construct a regular figure of three sides in or about a given \odot .

Problem 3.

To construct a circle passing through a given point and touching a given circle at a given point.



Let A be the given pt. outside the given \odot whose centre is O, and let P be the given pt. on the \odot^{ce} of the given \odot .

It is required to describe a circle passing through A and touching the given circle at P.

Cons. Join AP; and draw the \perp bisector of AP (*Prob. 2, Bk. I.*), meeting OP produced in C.

Then the \odot described with C as centre and CP as radius, will be the reqd. \odot .

Proof. Join CA.

BC being the \perp bisector of AP, $CA = CP$,

and \therefore the \odot described with C as centre and CP as radius, will also pass through A.

Also, since the two \odot s meet at P which is a pt. on the line joining their centres, \therefore the \odot s touch each other at P.

(*Th. 9, Note 2.*)

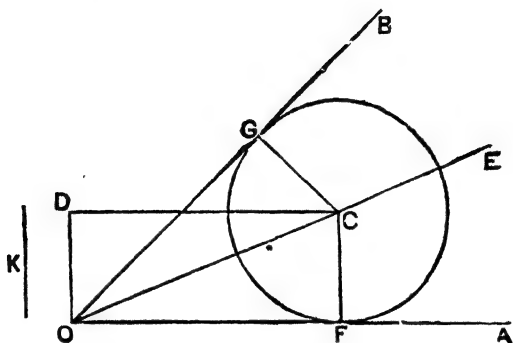
Hence, the \odot described with C as centre and CP as radius is the reqd. \odot .

Note. The solution of the present problem is discovered in the following manner :—*Assume* a circle to be described passing through A and also touching the given circle at P , and let C be its centre. Then, since the circles touch each other at P , OP produced must pass through C ; also, since $CA=CP$, C must be on the perpendicular bisector of AP . Hence C is the pt. where the perpendicular bisector of AP meets OP produced.

The process by which the solution of a problem is discovered is called **analysis**; whilst the opposite process, that of giving the necessary construction and proving its correctness, is called **synthesis**.

Problem 4.

To construct a circle, with a given radius, touching two given straight lines.



Let OA , OB be the two given str. lines intersecting at O , and let K be the given radius.

It is required to describe a circle which shall touch OA and OB , and have a radius equal to K .

Cons. Draw OE , the bisector of the $\angle AOB$.

(*Prob. 1, Bk. I.*)

Draw $OD \perp$ to OA , making it $= K$.

Through D , draw a str. line \parallel to OA meeting OE in C .

Draw CF , $CG \perp$ to OA , OB respectively. Then the \odot described with C as centre and CF as radius will be the reqd. \odot .

Proof. C being on the bisector of the $\angle AOB$,

$CF = CG$.

(*Bk. I, Th. 26.*)

\therefore the \odot described with C as centre and CF as radius will also pass through G .

Since OA is \perp to the radius CF at F ,

$\therefore OA$ touches the \odot at F . (Th. 7, Cor. 2.)

Similarly, OB touches the \odot at G .

Thus, OA and OB are both tangents to the \odot (1)

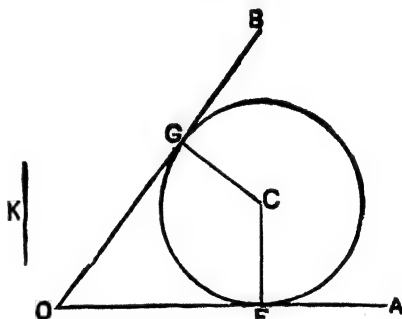
Again, since DF is a rectangle, by construction,

$\therefore CF = DO = K$ (2)

Hence, from (1) and (2), the circle described with C as centre and CF as radius is the reqd. \odot .

Note. The following is the *analysis* of the problem :—

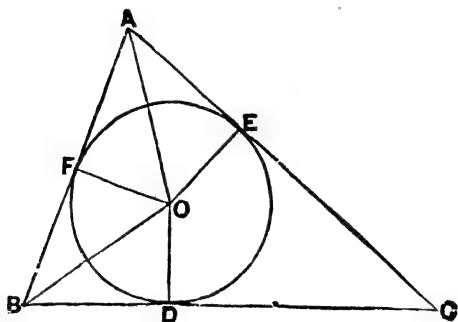
Assume a \odot to be described touching OA and OB at F and G respectively, and having its radius $= K$; and let C be the centre of the \odot . Join CF , CG . Then CF and CG , being radii drawn to the pts. of contact, are respectively \perp to OA and OB . (Th. 7.) Now, since $CF = CG$,



$\therefore C$ lies on the bisector of the $\angle AOB$. (Bk. I, Th. 26); and also since $CF = K$, $\therefore C$ lies on a str. line which is \parallel to OA and at a distance $= K$ from it. Hence, if these two lines be drawn, the pt. where they intersect will be the centre of the reqd. circle.

Problem 5. (Euc. IV. 4.)

To inscribe a circle in a given triangle.



Let ABC be the given \triangle .

It is required to describe a \odot which shall touch the sides of the $\triangle ABC$.

Cons. Draw the bisectors of the \angle s A and B , and let them meet at O .

Draw $OD, OE, OF \perp$ to BC, CA, AB respectively.

Then the \odot described with O as centre and OF as radius will be the reqd. \odot .

Proof. Since O is on the bisector of the $\angle A$,
 $\therefore OF = OE.$ (*Bk. 1. Th. 26.*)

Also, since O is on the bisector of the $\angle B$,
 $\therefore OF = OD.$

Hence, the \odot described with O as centre and OF as radius will also pass through E and D .

Again, since BC is \perp to the radius OD at D ,
 $\therefore BC$ touches the \odot at $D.$ (*Th. 7, Cor. 2.*)

Similarly, CA and AB touch the \odot at E and F respectively.

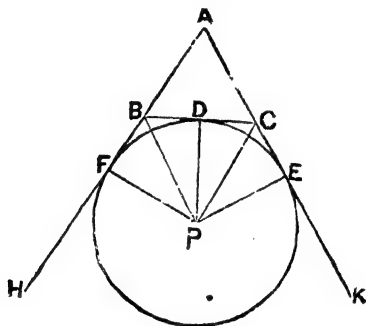
Hence the $\odot DEF$ is the reqd. \odot .

Analysis:—*Assume* the reqd. \odot to be described with its centre at O and touching BC, CA, AB at D, E, F respectively. Join OD, OE, OF . Then, since OD is the radius drawn to the pt. of contact of the tangent BC , $\therefore OD$ is \perp to BC , (Th. 7); similarly, OE, OF are \perp to CA, AB respectively. Now, since $OD = OF$, $\therefore O$ lies on the bisector of the $\angle B$; also, since $OE = OF$, $\therefore O$ lies on the bisector of the $\angle A$. (Bk. I, Th. 26). Hence, the pt. where these two bisectors meet must be the centre of the required circle.

Note. The circle inscribed in a triangle, the centre of the circle and its radius are respectively called the **in-circle**, the **in-centre** and the **in-radius** of the triangle.

Problem 6.

To construct a circle touching one side of a given triangle and the other two sides produced.



Let ABC be the given \triangle , and let the two sides AB , AC be produced to H , K respectively.

It is required to describe a \odot touching BC , BH and CK .

Cons. Draw the bisectors of the \angle s CBH and BCK , and let them meet at P .

Draw PD , PF , $PE \perp$ to BC , BH and CK respectively.

Then the \odot described with P as centre and PD as radius will be the required \odot .

Proof. Since P is on the bisector of the $\angle CBH$.

$$\therefore PD = PF. \quad (\text{Bk. I. Th. 26.})$$

Also, since P is on the bisector of the $\angle BCK$,

$$\therefore PD = PE.$$

Hence, the \odot described with P as centre and PD as radius will also pass through F and E .

Again, since BH is \perp to the radius PF at F .

$\therefore BH$ touches the \odot at F . (Th. 7, Cor. 2.)

Similarly, BC and CK touch the \odot at D and E respectively.

Hence the $\odot DEF$ is the reqd. \odot .

Note 1. The *Analysis* is left as an exercise for the student.

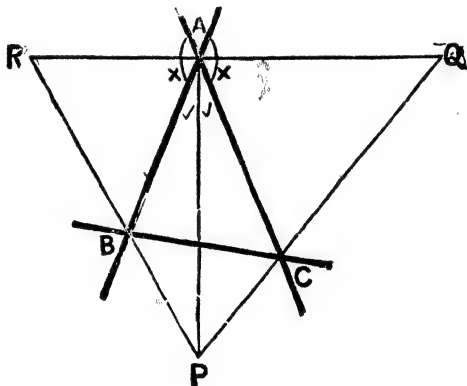
Note 2. Since $PF = PE$, the point P lies on the bisector of the $\angle A$. (Bk. I, Th. 26.)

Hence, *the bisector of one angle of a triangle and the bisectors of the exterior angles at the other two vertices are concurrent.*

Note 3. The circle which touches one side of a triangle and the other two sides produced is called an **escribed circle** of the triangle. The centre of an escribed circle of a triangle is called an **ex-centre** of the triangle.

Cor. If P, Q, R be the ex-centres of a $\triangle ABC$ opp. to A, B, C respectively, then PA, QB, RC are respectively \perp to QR, RP, PQ .

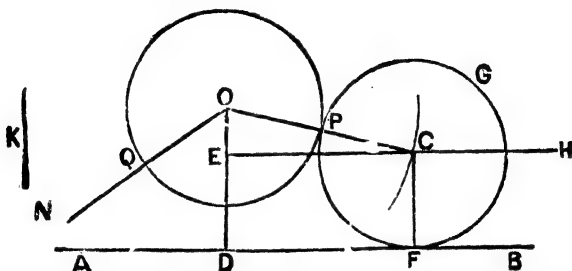
Let the bisectors of the exterior \angle s at A, B, C form the $\triangle PQR$, as in the adjoining diagram. Then P, Q, R are the ex-centres of the triangle, opposite to A, B, C respectively.



Join AP ; then AP is the bisector of the $\angle BAC$. Now the $\angle BAR =$ the $\angle CAQ$, (being halves of vertically opp. \angle s); also the $\angle PAB =$ the $\angle PAC$; hence the $\angle PAR =$ the $\angle PAQ$, and $\therefore PA$ is \perp to QR . Similarly QB is \perp to RP , and RC is \perp to PQ .

Problem 7.

To construct a circle, with a given radius touching a given circle and a given straight line.



Let the \odot with centre O be the given \odot , AB the given str. line, and K the given radius.

It is required to describe a circle which shall touch the circle (O) as well as the straight line AB , and have a radius $= K$.

Cons. Draw $OD \perp$ to AB , and from DO cut off $DE = K$; through E draw $EH \parallel$ to AB .

Take a radius OQ of the given \odot and produce it to N , making $QN = K$. Then ON is the sum of the radii of the given and the required \odot s.

With O as centre and ON as radius describe an arc cutting EH at C .

Join OC cutting the given \odot at P , and draw $CF \perp$ to AB .

Then the \odot described with C as centre and CP as radius will be the reqd. \odot .

Proof. Since $OC = ON$, of which $OP = OQ$,

$$\therefore CP = NQ = K;$$

also, since DC is a rectangle, by construction,

$$\therefore CF = ED = K.$$

Hence $CP = CF$; and \therefore the \odot described with C as centre and CP as radius will also pass through F .

Now, since the two \odot s meet at P which is a pt. on the line joining the centres.

\therefore the \odot s touch each other at P . (*Th. 9, Note. 2.*)

Also, since AB is \perp to the radius CF at F ,

$\therefore AB$ touches the \odot at F . (*Th. 7, Cor. 2.*)

Thus the \odot PFG touches both the given \odot and the given str. line, and has its radius $= K$;

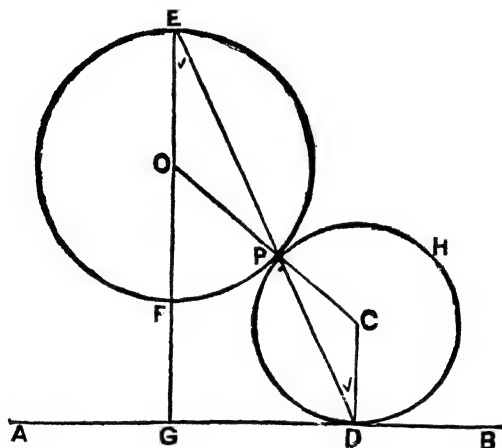
\therefore this is the required \odot . Q. E. F.

Note 1. The circle described with O as centre and ON as radius will also cut HE produced at some pt. With that point as centre another \odot may be constructed satisfying the conditions of the problem. This is left as an exercise for the student.

Note 2. If the reqd. \odot is *assumed* to be drawn, it is easy to see that the distance of its centre from O is equal to the sum of the radii, and the distance from $AB = K$. Hence follows the construction.

Problem 8.

To construct a circle touching a given circle and also touching a given straight line at a given point.



Let AB be the given str. line and D the given pt. in it, and let the \odot with centre O be the given \odot .

It is required to describe a circle touching the circle (O) and also touching AB at D .

Cons. Draw $OG \perp$ to AB cutting the \odot at F , and produce GO to meet the \odot in E .

(i) Join ED cutting the \odot at P ; join OP .

Through D draw a perpendicular to AB , meeting OP produced in C .

Then the \odot described with C as centre and CP as radius will be the reqd. \odot .

Proof. Since EG and CD are both \perp to AB ,

\therefore they are \parallel .

and \therefore the $\angle OEP =$ the $\angle CDP$;

also, the $\angle OPE =$ the $\angle CPD$. (*Vertically opp. \angle s.*)

But the $\angle OEP =$ the $\angle OPE$,

($\because OP = OE$),

\therefore the $\angle CDP =$ the $\angle CPD$,

and $\therefore CP = CD$.

Hence the \odot described with C as centre and CP as radius will also pass through D .

Again, since the two \odot s meet at P which is on the line joining the centres, \therefore the \odot s touch each other at P ;

(*Th. 9, Note. 2.*)

also, since AB is \perp to the radius CD at D ,

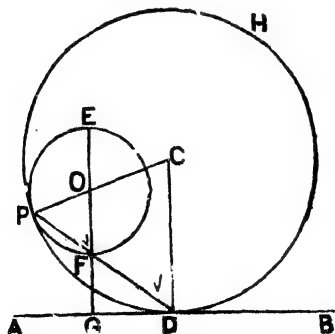
$\therefore AB$ touches the \odot at D . (*Th. 7, Cor. 2.*)

Thus, the $\odot PDH$ touches the given \odot , and also touches the given str. line at the given pt. D ;

\therefore this is the reqd. \odot .

Q. E. F.

(ii) Join DF and produce it to meet the \odot at P ; P . Join PO .



Through **D** draw a \perp to **AB**, meeting **PO** produced in **C**.

Then the \odot described with **C** as centre and **CP** as radius will be the required \odot .

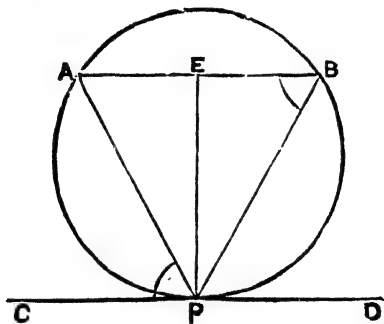
[The proof is left as an exercise for the student.]

Note 1. Let the required circle be *assumed* to be drawn, as in the first diagram; let **C** be its centre and **P** the pt. where it touches the given \odot . Then **OC** passes through **P**, and **CD** is \perp to **AB**. Join **DP** and produce it to meet the given \odot in **E**; join **EO**. Now, it is easily seen that the $\angle CDE = \text{the } \angle OED$; $\therefore \text{EO is } \parallel \text{CD}$, and $\therefore \text{EO produced cuts AB at rt. } \angle \text{s}$. We thus find that the pt. of contact of the two \odot s lies on the line **DE**, where **OE** is that radius of the given \odot which, when produced through **O**, cuts **AB** at rt. \angle s. The position of **P** being known, that of **C** is found at once, for it lies on **OP** produced and also on the \perp to **AB** through **D**.

Note 2. Let the reqd. circle be *assumed* to be drawn, as in the second diagram; let **C** be its centre and **P** the pt. where it touches the given \odot . Then **CO** produced passes through **P**, and **CD** is \perp to **AB**. Join **DP** cutting the given \odot at **F**; join **OF**. Now, it is easily seen that the $\angle CDP = \text{the } \angle OFP$; $\therefore \text{OF is } \parallel \text{to CD}$, and $\therefore \text{OF produced cuts AB at rt. } \angle \text{s}$. We thus find that the pt. of contact of the two \odot s lies on the line **DF** produced, where **OF** is that radius of the given \odot which, when produced through **F**, cuts **AB** at rt. \angle s. The position of **P** being known that of **C** is found at once, for it lies on **PO** produced and also on the \perp to **AB** through **D**.

Problem 9.

To construct a circle passing through two given points and touching a given straight line.



Let A , B be the two given pts. and CD the given str. line.

It is required to describe a circle passing through A and B , and also touching the line CD .

Join AB . Then AB is either \parallel to CD ; or meets CD when produced.

(i) Let AB be \parallel to CD .

Cons. Draw the \perp bisector of AB meeting CD in P .

Then the \odot passing through the pts. A , B , P will be the reqd. \odot .

Proof. Join PA , PB .

Now, P being on the \perp bisector of AB , $PA = PB$,

and \therefore the $\angle PBA =$ the $\angle PAB$.

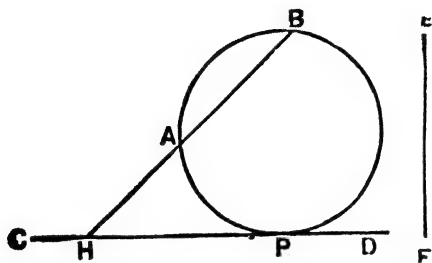
Since AB is \parallel to CD , \therefore the $\angle CPA =$ the $\angle PAB$.

Hence, the $\angle CPA =$ the $\angle PBA$, in the alt. segment ;

$\therefore CP$ touches the \odot at P . (*Th. 16, Cor.*)

Hence ABP is the reqd. \odot . Q. E. F.

(ii) Let BA produced meet CD in H .



CONS. Find a str. line EF the square on which is equal in area to the rect. $BH \cdot HA$, (*Bk. II. Prob. 3.*)

From HD cut off $HP = EF$.

Then the \odot passing through the pts. A, B, P will be the reqd. \odot .

PROOF. Since $HP^2 = HA \cdot HB$,

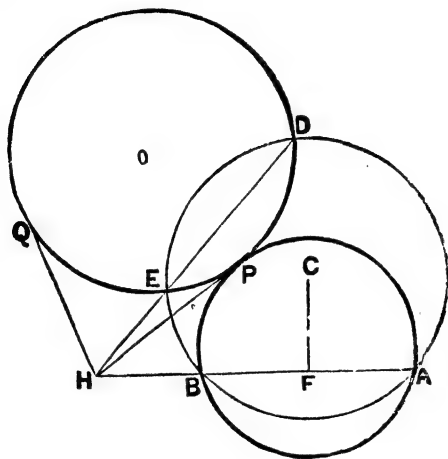
$\therefore HP$ touches the \odot at P . (*Th. 17, Cor. 2.*)

Hence ABP is the reqd. \odot . Q. E. F.

NOTE. HP may as well be cut off from HC produced. Thus there are two \odot s satisfying the given conditions.

Problem. 10.

To construct a circle passing through two given points and touching a given circle.



Let A, B be the two given pts. and let the \odot with centre O be the given \odot .

It is required to describe a circle passing through A and B , and also touching the circle (O) .

Join AB and draw FC , the \perp bisector of AB .

Then, FC may pass through O or may not.

(i) Let FC not pass through O .

CONS. With any pt. C on FC as centre, and CA or CB as radius, describe a \odot cutting the given \odot in D and E . Join DE and produce it to meet AB produced in H .

Draw HP to touch the given \odot at P . (*Th. 13, Note.*)

Construct a \odot through the pts. A, B, P ; then this will be the reqd. \odot .

Proof. Since the chords AB, DE of the \odot ABD intersect in H,

$$\therefore HA.HB = HD.HE. \quad (Th. 17.)$$

Again, since HP touches the given \odot at P and HD cuts it in D and E,

$$\therefore HP^2 = HD.HE. \quad (Th. 17, Cor. 1.)$$

Hence, $HA.HB = HP^2$,

and \therefore HP touches the \odot ABP at P. (*Th. 17, Cor. 2.*)

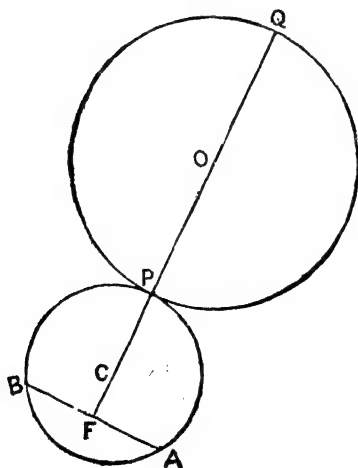
Now, the \odot ABP and the given \odot have a common tangent at P,

\therefore the two \odot s touch each other at P. (*Th. 9, Note 3.*)

Hence the \odot ABP is the reqd. \odot . Q. E. F.)

[Similarly, if the other tangent HQ to the given \odot be drawn, it may be proved as above that the \odot through A, B, Q is another \odot satisfying the given conditions.]

(ii) Let FC pass through O.



Cons. Let FO cut the given \odot at P .

Describe a \odot passing through the pts. A, B, P ; then this will be the reqd. \odot .

Proof. Since FP is the \perp bisector of AB ,

\therefore the centre of the $\odot ABP$ lies in FP .

Hence, P is a pt. on the line joining the centres of the two \odot s;

and since the \odot s meet at P ;

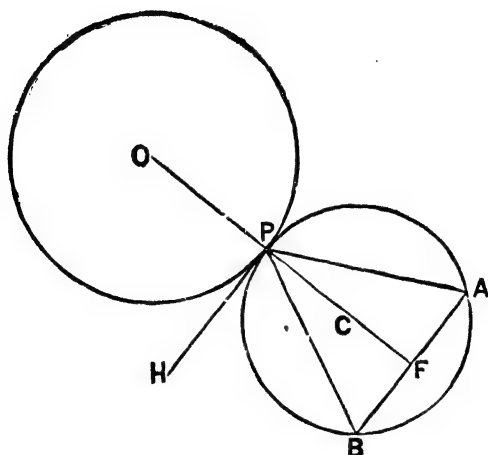
\therefore they touch each other at P . (*Th. 9, Note 2.*)

Hence the $\odot ABP$ is the reqd. \odot . Q. E. F.

[If FO be produced to meet the given \odot in Q , it may be proved as above, that the \odot through ABQ is another \odot satisfying the given conditions.]

Analysis :—Assume the required \odot to be drawn, touching the given \odot at P . Then the two \odot s have a common tangent at P ; let this tangent meet AB produced in H . $\therefore HP^2 = HA \cdot HB$, (*Th. 17, Cor. 1*). Through H let any secant be drawn cutting the given \odot in E and D ; then $HD \cdot HE = HP^2$ (*Th. 17, Cor. 1*). Hence $HA \cdot HB = HD \cdot HE$, which shows that the pts. A, B, E, D are concyclic (*Th. 17, Cor. 3*). Hence, if a \odot be constructed passing through A, B and also cutting the given \odot in two pts. D, E , then the pt. in which DE intersects AB is also the pt. in which the common tangent to the given and the reqd. \odot s intersects AB ; whence the construction follows.

If the common tangent at P be \parallel to AB , it will not meet AB produced.

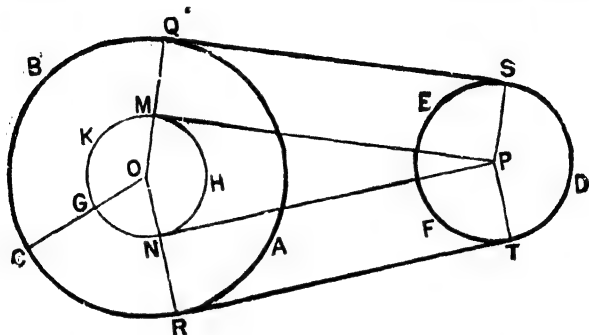


Join PA , PB . Since PH is \parallel to AB , \therefore the $\angle HPB =$ the $\angle PBA$; also the $\angle HPB =$ the $\angle PAB$, in the alt. segment.

Hence the $\angle PBA =$ the $\angle PAB$, and $\therefore PA = PB$. Hence P is on the \perp bisector of AB . The centre C of the $\odot APB$ also lies on the \perp bisector of AB . Hence both C and P lie on the \perp bisector of AB , and $\therefore PC$ produced bisects AB at rt. \angle s. But CP produced also passes through O . Hence OP produced bisects AB at rt. \angle s. In this case therefore the \perp bisector of AB passes through O and cuts the given \odot in the pt. at which the reqd. \odot touches it; whence follows the construction.

Problem 11.

To draw a common tangent to two given circles.



Let ABC and DEF be two given \odot s whose centres are O and P respectively, the radius of the first \odot being greater than the radius of the second.

It is required to draw a common tangent to the \odot s ABC and DEF .

CONS. Draw any radius OC of the $\odot ABC$.

(i) From CO cut off $CG =$ the radius of the $\odot DEF$.

With O as centre and OG as radius describe the $\odot GKH$.

Draw PM, PN tangents to the $\odot GKH$. (*Th. 13, Note.*)

Join OM, ON and produce them to meet the $\odot ABC$ at Q, R respectively.

In the $\odot DEF$, draw the radius $PS \parallel$ to OQ , and the radius $PT \parallel$ to OR .

Join QS, RT .

Then QS as well as RT are common tangents to the \odot s ABC and DEF .

Proof. Since PM touches the \odot GKH at M,

\therefore the \angle OMP is a rt. \angle ; (Th. 7.)

and \therefore the \angle PMQ is also a rt. \angle .

Now, OQ = OC, and of these the parts OM, OG are equal ; \therefore MQ = OG.

Hence PS = MQ ; and PS is also \parallel to MQ, by construction, \therefore PSQM is a parallelogram. (Bk. I, Th. 22.)

Also, since the \angle PMQ of this parallelogram is a rt. \angle .

\therefore all its \angle s are rt. \angle s.

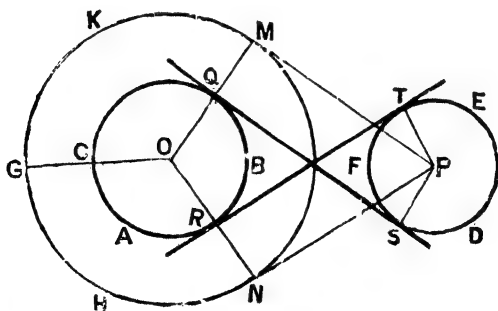
Thus, the \angle s OQS and PSQ are rt. \angle s ;

\therefore QS is a tangent to the \odot ABC as well as to the \odot DEF. (Th. 7, Cor. 2.)

In the same way it may be proved that RT is a tangent to both the given \odot s. Q. E. F.

(ii) From OC produced cut off CG = the radius of the \odot DEF.

With O as centre and OG as radius describe the \odot GKH,



Draw PM, PN tangents to the \odot GKH. (Th. 13, Note.)

Join OM, ON cutting the $\odot ABC$ at Q and R respectively.

In the $\odot DEF$, draw the radius PS \parallel to MO, and the radius PT \parallel to NO.

Join QS, RT.

Then QS as well as RT are common tangents to the $\odot s$ ABC and DEF.

[The proof is left as an exercise for the student.]

Note 1. It is clear that there are *four* common tangents to two given $\odot s$, if the $\odot s$ do not cut each other. A common tangent whose pts. of contact are on the same side of the line joining the centres is called a **direct common tangent**; whilst a common tangent whose pts. of contact are on the opposite sides of the line joining the centres is called a **transverse common tangent**. We have *direct common tangents* in the first diagram, and *transverse common tangents* in the second.

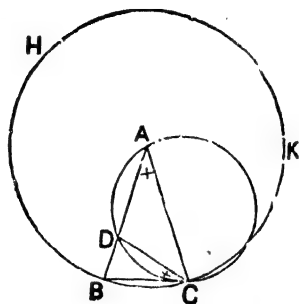
Note 2. If the direct common tangents be produced to meet at V, then each of the pts. P and O is equally distant from the lines QV and RV, and \therefore both of them lie on the bisector of the $\angle QVR$.

Hence OP produced passes through V. Thus, *the direct common tangents to two given $\odot s$ and the line joining their centres are concurrent.*

Note 3. In the second diagram, each of the pts. P and O is equally distant from TR and QS, and \therefore , if TR and QS intersect at W, both P and O lie on the bisector of the $\angle s$ TWS and QWR. Hence OP passes through W. Thus, *the transverse common tangents to two given $\odot s$ intersect on the line joining their centres.*

Problem 12. (Euc. IV. 10.)

To construct an isosceles triangle having each of the angles at the base double of the vertical angle.



Cons. Take any str. line AB and divide it at D so that $AB \cdot BD = AD^2$. (*Bk. II. Sec. IV. Prop. 7.*)

With A as centre and AB as radius describe the \odot HBK.

In this \odot place the chord $BC = AD$.

Join AC.

Then ABC is the \triangle reqd.

Proof. Join CD, and circumscribe a \odot about the \triangle ADC.

Since $BC^2 = AD^2$, $\therefore BC^2 = AB \cdot BD$.

\therefore BC touches the \odot ADC at C. (*Th. 17, Cor. 2.*)

Hence the $\angle BCD =$ the $\angle DAC$, in the alt. segment.

Hence, the $\angle BDC =$ the $\angle BCA$,

because each of them = the sum of the \angle s DAC, ACD.

But the $\angle ABC = \text{the } \angle BCA$;

($\because AC = AB$),

\therefore the $\angle ABC = \text{the } \angle BDC$.

Hence, $DC = BC = DA$;

and \therefore the $\angle DAC = \text{the } \angle DCA$.

Hence, DC is the bisector of the $\angle BCA$.

and \therefore the $\angle BCA = \text{twice the } \angle BCD$
 $= \text{twice the } \angle BAC.$

Hence the $\angle CBA$ also $= \text{twice the } \angle BAC.$

Thus, ABC is the required \triangle .

Q. E. F.

Analysis :—Assume ABC to be a \triangle such that each of its \angle s B and C is double the $\angle A$.

Let CD bisect the $\angle C$. Then the $\angle BCD = \frac{1}{2}$ the $\angle BCA = \text{the } \angle BAC$. Hence BC must touch the circum- \odot of the $\triangle ADC$ at C .

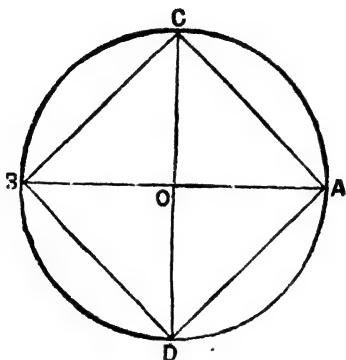
(Th. 16, Cor.)

$\therefore BC^2 = AB \cdot BD$. The $\angle DAC = \text{the } \angle DCA$, (because each of them is $\frac{1}{2}$ the $\angle BCA$), $\therefore DC = DA$; also, since the $\angle BDC = \text{the sum of the } \angle$ s $DAC, DCA = \text{twice the } \angle DAC = \text{the } \angle ABC$, $\therefore BC = DC$. Hence $BC = DA$, and $\therefore AB \cdot BD = AD^2$; which determines the position of D . Then the position of C also is determined, because the distance of C from $A = AB$, and its distance from $B = DA$.

Note. Evidently 5 times the $\angle A = \text{two rt. } \angle$ s. Hence the $\angle A = \text{one-fifth of two rt. } \angle$ s, and \therefore each of the \angle s $ABC, ACB = \text{two-fifths of two rt. } \angle$ s $= \text{one-fifth of four rt. } \angle$ s.

Problem 13. (EUC. IV. 6 AND 7.)

To construct a regular figure of four sides in or about a given circle.



Let the \odot with centre O be the given \odot .

It is required to construct an equilateral and equiangular quadrilateral in or about the \odot (\circ).

Cons. Take any diameter AB , and draw the diameter $CD \perp$ to AB .

(i) Join AC, CB, BD, DA .

Then $ABCD$ is a regular quadl. inscribed in the given \odot .

Proof. The four \angle s at O are equal, each of them being a rt. \angle ;

\therefore the four arcs AC, CB, BD, DA are equal.

(Th. 4, Cor. 1.)

Hence the chords AC, CB, BD, DA are equal;

(Th. 5, Cor. 1.)

i.e., the quadrilateral ACBD is equilateral.

Again, the $\angle ACB$ being in a semi- \odot , is a rt. \angle ; and similarly, each of the \angle s at B, D, A is a rt. \angle .

\therefore all the \angle s of the quadl. are equal. Thus the quadl. ACBD is both equilateral and equiangular, and it is also *inscribed* in the given \odot .

(ii) Draw tangents to the \odot at A, C, B, D forming the quadl. PQRS.

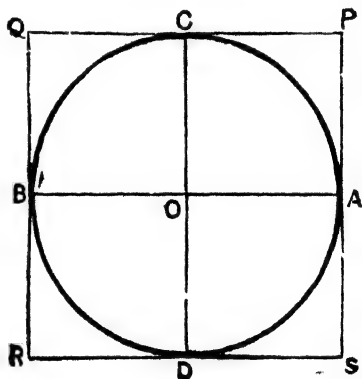
Then PQRS is a regular quadl. circumscribed about the given \odot .

Proof. The \angle s at A, C, B, D are all rt. \angle s. (*Th. 7.*)

Hence the $\angle OAP =$ the $\angle DOA$;

\therefore PS is \parallel to CD. }
Similarly, QR is \parallel to CD. }

Hence PS, CD, QR are \parallel to one another ;



and similarly QP, BA, RS are \parallel to one another.

Now, since BP is a par^m.

\therefore PQ = AB ; }
and similarly, QR = CD. }

\therefore PQ = QR.

Similarly, QR = RS, and RS = SP.

Again, since CA is a par^m .,

\therefore the $\angle APC =$ the $\angle AOC =$ a rt. \angle ;

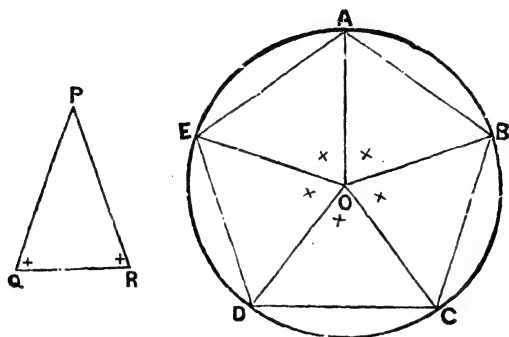
similarly, each of the \angle s at Q, R, S is a rt. \angle .

Hence the quadl. $PQRS$ is also equiangular. Thus the quadl. $PQRS$ is both equilateral and equiangular. and it is also *circumscribed* about the given \odot . Q. E. F.

Note. It is easy to see that a regular quadrilateral is a square. In the diagrams of Problem 13, each of the quadls. $ACBD$ and $PQRS$ is a rectangle with adjacent sides equal, and is \therefore a square.

Problem 14. (Euc. IV. 11 AND 12.)

To construct a regular figure of five sides in or about a given circle.



Let the \odot with centre O be the given \odot .

It is required to construct an equilateral and equiangular pentagon in or about the circle (O).

Cons. Construct an isosceles $\triangle PQR$ having each of the \angle s at Q and R double of the \angle at P . (*Prob. 12.*)

Draw any radius OA , and then draw the radius OB making the $\angle AOB =$ the $\angle Q$.

Now, successively draw the radii OC, OD, OE making the \angle s BOC, COD, DOE each $=$ the $\angle Q$.

(i) Join AB, BC, CD, DE, EA . Then $ABCDE$ is a regular pentagon inscribed in the given \odot .

Proof. Each of the \angle s at Q, R is one-fifth of four rt. \angle s. (*Prob. 12, Note.*)

Hence each of the \angle s AOB, BOC, COD, DOE is one-fifth of four rt. \angle s;

\therefore the $\angle EOA$, which together with those four \angle s makes up the four rt. \angle s at O , must also be = one-fifth of four rt. \angle s.

Thus the five \angle s at O are equal ;

\therefore the five arcs AB, BC, CD, DE, EA are equal.
(Th. 4, Cor. 1.)

Hence the five chords AB, BC, CD, DE, EA are equal.
(Th. 5, Cor. 1.)

The pentagon $ABCDE$ is \therefore equilateral.

Again, the arc $AEDC$ = the arc $BAED$;

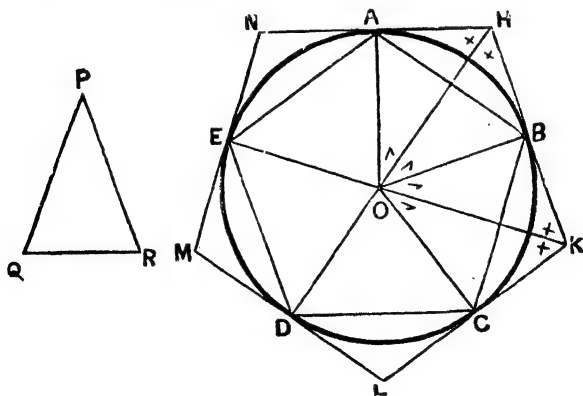
(\because each of them = 3 times the arc AB),

\therefore the $\angle ABC$ = the $\angle BCD$. (Th. 10, Cor. 2.)

Similarly, every two consecutive \angle s of the pentagon are equal.

Hence the pentagon $ABCDE$ is both equilateral and equiangular, and it is also *inscribed* in the given \odot .

(ii) Draw tangents to the \odot at A, B, C, D, E forming the pentagon $HKLMN$. Then $HKLMN$ is a regular pentagon circumscribed about the \odot .



Proof. Join OH, OK .

It may be proved, as before, that the five radii OA, OB, OC, OD, OE divide the four rt. \angle s at O into five equal parts.

Also the \angle s at A, B, C, D, E are rt. \angle s. (Th. 7.)

Now, since the tangents AH, BH meet at H,

$$\begin{array}{ll} \therefore AH = BH, & \\ \text{the } \angle AOH = \text{the } \angle BOH, & \\ \text{and the } \angle OHA = \text{the } \angle OHB. & \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{(Th. 8.)} \\ \text{Hence, the } \angle BOH = \frac{1}{2} \text{ the } \angle AOB, & \\ \text{and the } \angle OHB = \frac{1}{2} \text{ the } \angle AHB. & \left. \begin{array}{l} \\ \\ \end{array} \right\} \\ \text{Similarly, the } \angle BOK = \frac{1}{2} \text{ the } \angle BOC, & \\ \text{and the } \angle OKB = \frac{1}{2} \text{ the } \angle BKC. & \left. \begin{array}{l} \\ \\ \end{array} \right\} \end{array}$$

In the \triangle s OBH, OBK, we have

$$(1) \quad \text{the } \angle OBH = \text{the } \angle OBK, \quad (\text{rt. } \angle \text{s.})$$

$$(2) \quad \text{the } \angle BOH = \text{the } \angle BOK. \quad (\text{halves of the equal } \angle \text{s } AOB, BOC)$$

and (3) the side OB common;

\therefore the two \triangle s are congruent.

$$\text{Hence} \quad BH = BK \quad \dots \dots \dots (a)$$

$$\text{and the } \angle OHB = \text{the } \angle OKB \quad \dots \dots \dots (\beta)$$

From (a), $HK = 2BH$; and in the same way it may be proved that $NH = 2AH$.

$$\therefore NH = HK, (\because AH = BH);$$

similarly $HK = KL, KL = LM, LM = MN$, and $MN = NH$.

Hence the pentagon HKLMN is equilateral.

From (β), halves of the \angle s AHB and BKC are equal,

\therefore the $\angle AHB =$ the $\angle BKC$;

similarly, every two consecutive \angle s of the pentagon are equal.

Hence the pentagon $HKLMN$ is also equiangular.

Thus the pentagon $HKLMN$ is both equilateral and equiangular, and it is also *circumscribed* about the given \odot .

Q. E. F.

Note. An angle of an equilateral Δ is one-third of two rt. \angle s, and \therefore one-sixth of four rt. \angle s. Hence, the four rt. \angle s at the centre of a \odot may be divided into six equal parts, in the manner exemplified in the present problem; which enables us to construct a regular figure of six sides in or about the circle. This is left as an exercise for the student.

EXERCISE (26)

1. Construct a circle passing through a given point and touching a given straight line at a given point.

2. Construct a \odot , with a given radius, passing through two given pts.

3. Construct a circle, with a given radius, which shall touch a given straight line and pass through a given point.

4. Construct a circle, with a given radius, touching two given circles.

5. Two parallel straight lines are cut by a third straight line. Construct a circle touching each of these three straight lines.

6. Find the magnitude of the angle subtended at the centre of a given circle by each of the sides of an inscribed equilateral triangle. Hence shew how to construct an equilateral triangle in a given circle.

7. Prove that tangents drawn to a circle at the vertices of an inscribed equilateral triangle will form another equilateral triangle. Hence, given a circle, shew how to construct an equilateral triangle about it.

8. Divide the circumference of a given circle into six equal parts. Prove that the chord joining any two consecutive points of division is equal to the radius of the circle.

9. Construct a regular figure of six sides in a given circle.

10. Construct a regular figure of six sides about a given circle.

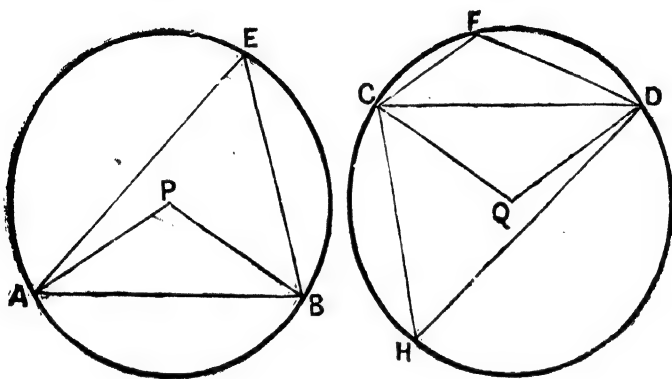
11. Construct an angle of 120° , and also an angle of 72° , and hence construct an angle of 24° .

12. Construct a regular figure of fifteen sides in a given circle.

SECTION IV.

MISCELLANEOUS PROPOSITIONS

1. *If two triangles standing on equal bases have vertical angles that are either equal or supplementary then their circum-circles are equal.*



Let the \triangle s AEB, CFD standing on equal bases AB, CD have their vertical \angle s AEB, CFD *supplementary*.

Let P, Q be the centres of the \odot s circumscribed about the \triangle s AEB, CFD respectively.

To prove that these two circles have equal radii.

Proof. Join PA, PB, QC, QD,

Take any pt. H on the arc conjugate to the arc CFD.
Join CH, DH.

Then the \angle CHD = the \angle AEB,

each of them being supplementary to the \angle CFD ;

\therefore the $\angle CQD =$ the $\angle APB$, (doubles of equal \angle s).

Now, since the $\angle PAB =$ the $\angle PBA$, each of them
 $= \frac{1}{2}$ their sum $= \frac{1}{2}$ the supplement of the $\angle APB$;

similarly, each of the \angle s QCD, QDC

$= \frac{1}{2}$ the supplement of the $\angle CQD$.

Hence, each of the \angle s $PAB, PBA =$ each of the
 \angle s QCD, QDC .

Then the two isos. \triangle s PAB, QCD have their bases
 equal, and also the \angle s at the base of the one $=$ the \angle s at
 the base of the other ;

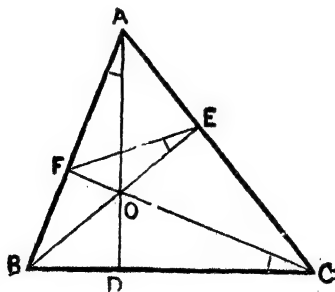
\therefore these two \triangle s are congruent. (Bk. I, Th. 18.)

Hence, each of the sides $PA, PB =$ each of the sides
 QC, QD .

Thus, the radii of the two \odot s are equal ; which proves
 the proposition.

Note. It is obvious, from the foregoing demonstration, how to
 deal with the case when the two triangles have their vertical angles
equal.

2. *The perpendiculars drawn from the vertices of a
 triangle to the opposite sides are concurrent.*



Let BE, CF be \perp s to the sides CA, AB respectively of the $\triangle ABC$; and let them intersect at O .

Join AO and produce it to meet BC at D .

To prove that AD is \perp to BC .

Proof. Join FE .

The \angle s OFA, OEA are supplementary, each of them being a rt. \angle ;

\therefore the quadl. $AFOE$ is cyclic.

\therefore the $\angle FAO =$ the $\angle EFO$, in the same segment. (α)

Again, the $\angle BFC =$ the $\angle BEC$, each being a rt. \angle ;

\therefore the four pts. B, F, E, C are concyclic, (*Th. 12.*)

\therefore the $\angle FEB =$ the $\angle FCB$, in the same segment. (β)

Hence, from (α) and (β),

the $\angle BAD =$ the $\angle BCF$.

Now in the \triangle s BAD, BCF the \angle at B is common, and the $\angle BAD =$ the $\angle BCF$;

\therefore the remaining $\angle ADB =$ the remaining $\angle CFB$ = a rt. \angle .

Hence, AD is \perp to BC .

Q. E. D.

Note 1. For an alternative proof, see Book I., Sec. VIII, Prop. 3.

Note 2. The point of intersection of the perpendiculars from the vertices of a triangle to the opposite sides is called the **ortho-centre** of the triangle. Thus, in the preceding diagram, the point O is the *ortho-centre* of the triangle ABC .

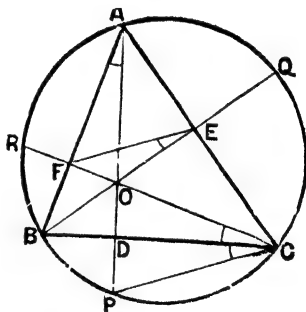
Note 3. The triangle formed by joining the feet of the perpendiculars drawn from the vertices of any given triangle to the opposite sides is called the **pedal triangle** of the given triangle. In the preceding diagram, the triangle DEF is the *pedal triangle* of the triangle ABC .

Cor. 1. If O is the ortho-centre of a triangle ABC , the circum-circles of the four triangles ABC , BOC , COA , AOB are equal to one another.

In the preceding diagram, the quadl. $AFOE$ being cyclic, the $\angle FAE$ is supplementary to the $\angle FOE$ and is \therefore supplementary to the $\angle BOC$. Thus, the Δ s BAC , BOC stand on the same base and have their vertical \angle s supplementary;

\therefore the circum- \odot of the ΔBOC is = that of the ΔABC (*Prop 1.*). Similarly, the circum- \odot of each of the Δ s COA , AOB is = that of the ΔABC . Hence the four circum- \odot s are equal to one another.

Cor. 2. If the perpendiculars AD , BE , from the vertices A , B , C of a triangle to the opposite sides meet at O and if AD , BE , CF be produced to meet the circum-circle of the triangle ABC in P , Q , R respectively, then D , E , F are the mid-points of OP , OQ , OR respectively.



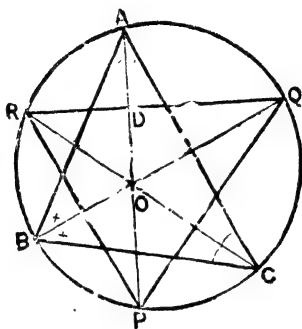
Join PC . Then the $\angle BCP =$ the $\angle BAP =$ the $\angle BCF$.

Now the Δ s ODC , PDC are evidently congruent.

(*Bk. I, Th. 18.*)

$\therefore OD = DP$. Similarly, $OE = EQ$ and $OF = FR$.

3. If O is the in-centre of a triangle ABC , and if AO, BO, CO be produced to meet the circum-circle in P, Q, R respectively, then O is the ortho-centre of the triangle PQR .



Proof. Let AP intersect RQ at D .

AB, BC, CA are tangents to the in- \odot of the $\triangle ABC$;

$\therefore AO, BO, CO$ are the bisectors of the
 \angle s A, B, C . (Th. 8.)

Now, the $\angle PRC =$ the $\angle PAC$ (Th. 11.)
 $= \frac{1}{2}$ the $\angle A$;

the $\angle CRQ =$ the $\angle CBQ = \frac{1}{2}$ the $\angle B$;
 and the $\angle RPA =$ the $\angle RCA = \frac{1}{2}$ the $\angle C$.

Hence, in the $\triangle PRD$, we have

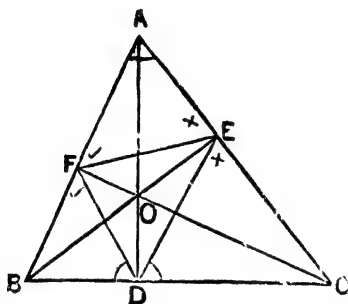
the $\angle PRD +$ the $\angle RPD = \frac{1}{2}$ the $\angle A + \frac{1}{2}$ the $\angle B + \frac{1}{2}$
 the $\angle C =$ one rt. \angle ;

\therefore the remaining $\angle PDR =$ one rt. \angle , and $\therefore PA$ is
 \perp to QR .

Similarly QB is \perp RP , and RC is \perp to PQ .

Hence, O is the ortho centre of the $\triangle PQR$. Q. E. D

4. *In an acute-angled triangle the perpendiculars drawn from the vertices to the opposite sides bisect the angles of the pedal triangle.*



Let O be the ortho-centre, and DEF the pedal Δ , of the acute-angled ΔABC .

To prove that AD , BE , CF bisect the \angle s D , E , F respectively of the ΔDEF .

Proof. The $\angle ADB =$ the $\angle AEB$, each being a rt. \angle .

\therefore the quadl. $AEDB$ is cyclic (Th. 12.)

Hence, the $\angle EDC =$ the $\angle BAE$; (Th. 14, Cor. 1.)

and similarly, the $\angle FDB =$ the $\angle CAF$.

\therefore the $\angle EDC =$ the $\angle FDB$;

and the whole $\angle ADC =$ the whole $\angle ADB$.

\therefore the remaining $\angle ADE =$ the remaining $\angle ADF$.

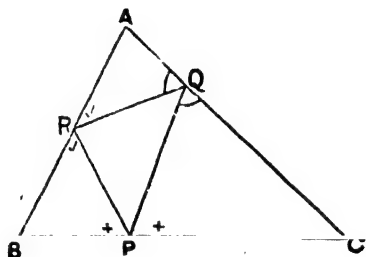
Thus, AD bisects the $\angle EDF$.

Similarly, BE and CF bisect the \angle s DEF and DFE respectively.

Q. E. D.

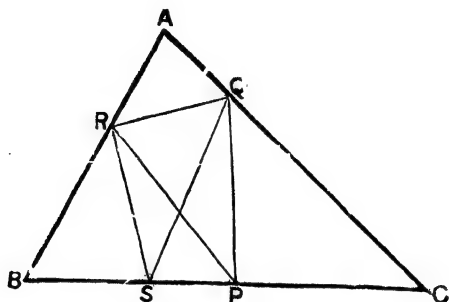
Note. The pedal triangle of an acute-angled triangle is evidently one inscribed in that triangle; and it is also clear, from the preceding diagram that the pedal triangle possesses the property of having every two of its sides equally inclined to the side of the given triangle on which they meet. That, of all triangles inscribed in a given triangle, it is the pedal triangle alone that possesses this property, can be proved as follows:—

Let the $\triangle PQR$ inscribed in the acute-angled $\triangle ABC$ be such that its sides RP , PQ are equally inclined to BC , the sides PQ , QR are equally inclined to CA , and the sides QR , RP are equally inclined to AB , as shewn in the accompanying diagram. Now, the six \angle s marked at P , Q , R , to-



gether with the \angle s at A , B , C make up six rt. \angle s; hence, those six \angle s are together = 4 rt. \angle s. Hence the three \angle s AQR , ARQ and QPC (one being taken from each pair) are together = 2 rt. \angle s; and \therefore the $\angle QPC =$ the $\angle A$, which shews that the quadl. $ABPQ$ is cyclic. Similarly, each of the quadls. $BCQR$ and $CARP$ is cyclic. Hence, if AP , BQ and CR be joined, it is easy to see that the $\angle APQ =$ the $\angle ABQ =$ the $\angle RCQ =$ the $\angle APR$; whence the $\angle APC =$ the $\angle APB$; and $\therefore AP$ is \perp to BC . Similarly BQ is \perp to CA and CR is \perp to AB . Hence the $\triangle PQR$ is no other than the pedal triangle. Thus, of all the \triangle s that can be inscribed in the $\triangle ABC$, it is only the pedal \triangle that has the above property.

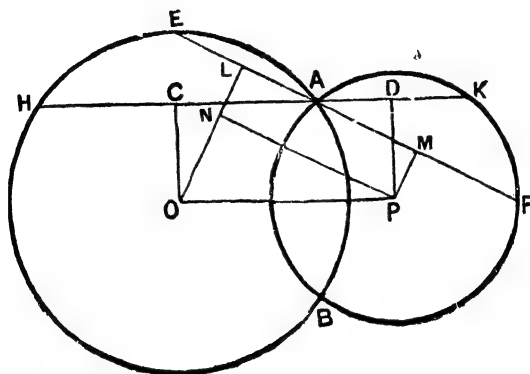
Cor. Of all triangles that can be inscribed in an acute-angled triangle, the pedal triangle is that which has the least perimeter.



Let ABC be an acute-angled \triangle ; then, of all the \triangle s that can be inscribed in it, *either the pedal \triangle , or some one of the others*, must have the least perimeter. Let PQR be *any* of the inscribed \triangle s *other than* the pedal \triangle , as in the above diagram. Since PQR is not the pedal \triangle , it must have *at least* one pair of sides that are *not* equally inclined to that side of the $\triangle ABC$ on which they meet ; let RP, PQ be that pair. Suppose then that S is the pt. in BC such that RS, SQ are equally inclined to it ; then $RS + SQ$ is $< RP + PQ$, (*Blk. I, Sec. VIII, Prop. 8.*) and \therefore the perimeter of the $\triangle RSQ$ is $<$ that of the $\triangle RPQ$. The $\triangle PQR$ is \therefore *not* one having the least perimeter. Hence, the \triangle of minimum perimeter is clearly *no other than* the pedal \triangle ; or, which is the same thing, it is the pedal \triangle that has the minimum perimeter.

5. Two circles whose centres are O and P cut each other at the points A and B ; then, of all the lines drawn

through A or B and terminated by the two circumferences, the greatest is that which is parallel to OP.



Let HK be the line through A \parallel to OP, and let EF be any other line through A, as in the above diagram.

To prove that HK is $>$ EF.

Proof. Draw OC, PD \perp to HK; OL, PM \perp to EF, and PN \perp to OL.

Since OC is \perp to HA,

$$\therefore HA = 2AC. \quad (\text{Th. 1.})$$

Similarly, $AK = 2AD$;

$$\therefore HK = 2CD. \}$$

In the same way, $EF = 2LM. \}$

Now CP and NM are evidently rectangles;

$$\therefore CD = OP, \}$$

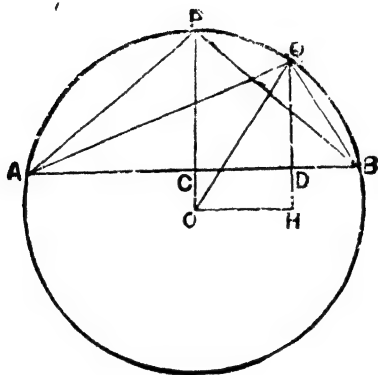
$$\text{and } LM = NP. \}$$

But, in the rt. $\angle d \triangle OPN$,

the hypotenuse OP $>$ the side NP;

$$\therefore CD > LM, \text{ and } \therefore HK > EF. \quad \text{Q. E. D.}$$

6. *Of all triangles standing on the same base and having the same vertical angle, the isosceles is that which has the greatest area and the greatest perimeter.*



Let the isos. $\triangle APB$ stand on the base AB ; and let AQB be *any other* \triangle on the *same* base, having the vertical $\angle AQB = \text{the } \angle APB$.

(i) To prove that the area of the $\triangle APB$ is greater than the area of the $\triangle AQB$.

Proof. Since the $\angle APB = \text{the } \angle AQB$,

\therefore the four pts. A, P, Q, B are concyclic. (*Th. 12.*)

Let O be the centre of the \odot passing through these four points, and draw $OC, QD \perp$ to AB .

Then OC is the \perp bisector of AB ; (*Th. 1.*)

and since $PA = PB$, $\therefore OC$ produced must pass through P .

Draw $OH \perp$ to QD produced; join OQ .

Now, in the rt. \angle d $\triangle OQH$, the hypotenuse $OQ > QH$.

Hence $OP > QH$

and, CH being a rectangle, $CO = QH$;

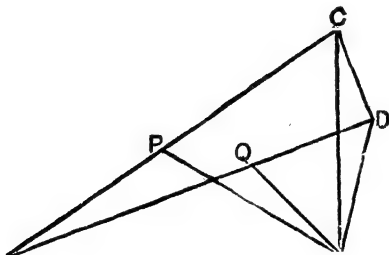
$\therefore PC > QD$.

Now, the area of the $\triangle APB = \frac{1}{2}$ the rect. $AB \cdot PC$;

and that of the $\triangle AQB = \frac{1}{2}$ the rect. $AB \cdot QD$.

Hence the area of the $\triangle APB >$ the area of the $\triangle AQB$.

(ii) To prove that the perimeter of the $\triangle APB$ is greater than the perimeter of the $\triangle AQB$.



Proof. Produce AP to C making $PC = PB$, and AQ to D making $QD = QB$.

Join CB , DB , CD .

Since $AP = PB = PC$, \therefore the \odot described on AC as diameter will pass through B ;

\therefore the $\angle ABC$ is a rt. \angle .

Now, since $PC = PB$, \therefore the $\angle PBC =$ the $\angle PCB$,

and \therefore the $\angle APB =$ double the $\angle ACB$;

similarly, the $\angle AQB =$ double the $\angle ADB$.

Hence, the $\angle ACB =$ the $\angle ADB$;

and \therefore the quadl. $ACDB$ is cyclic.

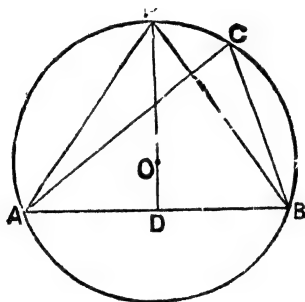
(Th. 12.)

Hence, the $\angle ADC = \text{the } \angle ABC$ (Th. 11.)
 $= \text{a rt. } \angle$.

Now, in the rt. \angle d $\triangle ADC$,
 the hypotenuse $AC > \text{the side } AD$;
 $\therefore AP + PB > AQ + QB$.

Hence the perimeter of the $\triangle APB$ is $>$ the perimeter of the $\triangle AQB$. Q. E. D.

7. *Of all triangles that can be inscribed in a given circle, that which has the greatest area or the greatest perimeter is equilateral.*



Let ABC be the given \odot of which the centre is O .

(i) To prove that of all the \triangle s that can be inscribed in the $\odot ABC$, that which has the greatest area is equilateral.

Proof. The \triangle which has the greatest area must be *either* equilateral or non-equilateral.

Let ABC be *any* non-equilateral \triangle inscribed in the given \odot .

It must then have *at least* one pair of unequal sides ; let AC, CB be that pair.

Draw $OD \perp$ to AB , and produce DO to meet the \odot in P ; join PA , PB .

Now, PD is the \perp bisector of AB (Th. 1.)

$\therefore PA = PB$.

Hence, of the two \triangle s APB , ACB which stand on the same base AB and have equal vertical \angle s, the $\triangle APB$ is isosceles.

Hence the area of the $\triangle APB$ is $>$ that of the $\triangle ACB$ (Last Prop.)

the $\triangle ACB$ \therefore has *not* the maximum area.

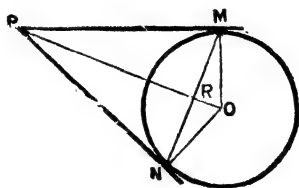
Hence, the \triangle of maximum area inscribed in the \odot can *not* be non-equilateral;

\therefore it must be equilateral.

(ii) To prove that of all the \triangle s that can be inscribed in the \odot ABC , that which has the greatest perimeter is equilateral.

[The Proof is similar to the above and is left as an exercise for the student.]

8. P is any point outside a circle of which the centre is O ; PM , PN are tangents to the circle, and MN the chord of contact. If OP intersects MN at R , then $OR \cdot OP = OM^2$.



Proof. Since the \angle s OMP , CNP are supplementary, each of them being a rt. \angle ,

\therefore the quadl. $OMPN$ is cyclic. (Th. 15.)

Hence, the $\angle OMN = \text{the } \angle OPN$ (Th. 11.)
 $= \text{the } \angle OPM.$ (Th. 8.)

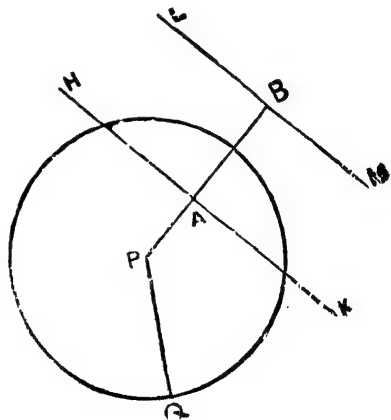
Hence, OM touches at M the \odot circumscribed about the $\triangle PMR$; (Th. 16, Cor.)

$\therefore OR \cdot OP = OM^2.$ (Th. 17, Cor. 1.)

Note 1. PM being $= PN$ (Th. 8), the \perp bisector of MN passes through P ; for a similar reason, the \perp bisector of MN passes through O . Hence OP is the \perp bisector of MN . Thus, *the chord of contact of tangents drawn to a circle from an external point is bisected perpendicularly by the line joining that point to the centre.*

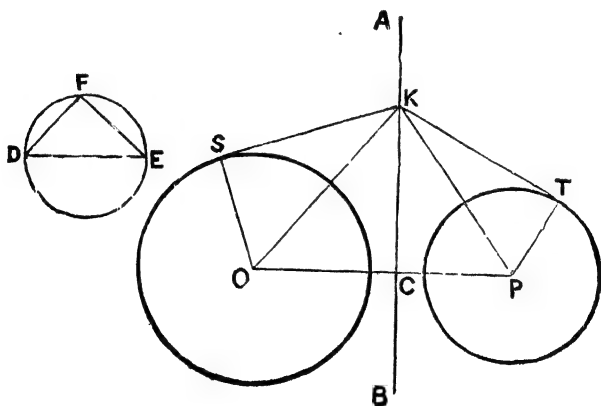
Note 2. If two points are so related to a circle that they lie in a straight line passing through the centre, and the rectangle contained by their distances from the centre is equal to the square on the radius, then each of these points is said to be the **inverse** of the other, with respect to the given circle. Thus, in the preceding diagram, the points R and P are a pair of *inverse points* with respect to the circle whose centre is O and radius $= OM$.

Note 3. If A and B are a pair of inverse points with respect to a circle whose centre is P and radius $= PQ$ (that is, if P, A, B are collinear and $PA \cdot PB = PQ^2$), and if through A a straight line HK be drawn \perp to the line PAB , then the point B is said to be the **pole** of the line HK , and the line HK , is said to be the **polar** of the point B . Also, if LM be drawn through B perpendicular to the line PAB , then A is the *pole* of the line LM , and the line LM is the *polar* of the point A . Thus, in the preceding diagram, the point P is the *pole* of the line MN , and MN is the *polar* of the point P .



Note 4. Two circles are said to be **orthogonal**, or to intersect **orthogonally**, when the two tangents at either point of intersection are perpendicular to each other. In the diagram on page 275, M is a pt. of intersection of the given \odot and the \odot about the $\triangle PMR$; also MP and MO , which are the tangents at M to those two \odot s, respectively, are \perp to each other. Hence the given \odot and the \odot about the $\triangle PMR$ intersect *orthogonally*. Similarly, the given \odot and the \odot about the $\triangle PNR$ intersect *orthogonally*.

Q. O and P are the centres of two given circles, the line OP being greater than the sum of their radii. Divide OP into two parts such that the difference of the squares on them may be equal to the difference of the squares on the radii of the circles. If C be the point of division, and if a straight line be drawn through C perpendicular to OP , prove that every point on this straight line is such that tangents from it to the two circles are equal.



(i) Let the \odot with centre O have the large radius.

Cons. Take a str. line DE = the radius of the circle (O); on DE as diameter describe a \odot , and in it place the chord EF = the radius of the circle (P); join DF .

Then the $\angle DFE$, being in a semi- \odot , is a rt. \angle ; and we have $DF^2 = DE^2 - EF^2$

= the difference of the sqs. on the radii.

Now, divide OP at C such that $OC^2 - CP^2 = DF^2$.
(*Blk. II, Sec. IV. Prop. 4.*)

Then C is the reqd. pt.

(ii) Let AB be the str. line drawn through $C \perp$ to OP ; on AB take any pt. K .

Let KS and KT be tangents to the \odot s (O) and (P) respectively.

To prove that $KS = KT$.

Proof. Join OS , PT , OK , PK .

Then the \angle s OSK and PTK are right. (*Th. 7.*)

$$\begin{array}{l} \text{Now,} \quad OK^2 = OC^2 + CK^2, \\ \text{and} \quad PK^2 = PC^2 + CK^2. \end{array}$$

$$\begin{aligned} \therefore OK^2 - PK^2 &= OC^2 - PC^2 \\ &= DF^2 \\ &= OS^2 - PT^2 \end{aligned}$$

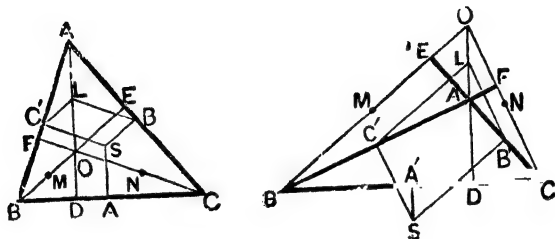
$$\text{Hence, } OK^2 - OS^2 = PK^2 - PT^2.$$

$$\text{Hence, } KS^2 = KT^2, \text{ and } \therefore KS = KT. \quad \text{Q. E. D.}$$

Note 1. The str. line AB , which has the property proved above, is called the **radical axis** of the two given circles.

Note 2. From Note to Cor. 1, Th. 17, it is clear that when two circles intersect, their common secant satisfies the condition of the radical axis, obviously with the exception of that portion which forms the common chord of the two circles.

10 *The distance of each vertex of a triangle from the or. o-centre is double the distance of the circum-centre from the opposite side.*



Let O be the ortho-centre of the $\triangle ABC$; AD , BE , CF being the \perp s from A , B , C on the opposite sides.

Let S be the circum-centre of the $\triangle ABC$, and A' , B' , C' the mid-pts. of the sides opposite to A , B , C respectively.

Then SA' , SB' , SC' are \perp s respectively to BC , CA , AB .
(Th. 1.)

To prove that $AO = 2SA'$, $BO = 2SB'$, $CO = 2SC'$.

Proof. SA' is \parallel to AD , because both of them are \perp to BC ;

Similarly, SB' is \parallel to BE , and SC' is \parallel to CF . Let L , M , N be the mid-pts. of AO , BO , CO respectively.

Join LB' , LC' .

Now, $B'L$ passes through the mid-pts. of CA and OA ;

$\therefore B'L$ is \parallel to CO and \therefore to SC' ;
similarly, $C'L$ is \parallel to BO and \therefore to SB' . }

Hence the quadl. SL is a par^m . ;

$\therefore SB' = C'L = \frac{1}{2} BO$,
and $SC' = B'L = \frac{1}{2} CO$. }

Similarly, by joining MC' , MA' , (or NA' , NB'), it can be shewn that $SA' = \frac{1}{2}AO$. Q. E. D.

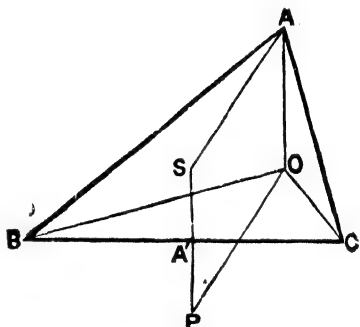
Note. If O is the ortho-centre of any given $\triangle ABC$, the following constructions for the circum-centre should be carefully noted. Through A' , the mid-pt. of the side opp. to A , draw $AS \parallel$ to, and in the same *sense* as, OA ; make $AS = \frac{1}{2}OA$. Then S is the circum-centre of the given \triangle .

Cor. 1. If O is the ortho-centre of any triangle ABC , then the circum-circles of the four triangles ABC , BOC , COA , AOB are equal to one another.

It is easy to see that the pts. A, B, C are the ortho-centres of the \triangle s BOC , COA , AOB respectively.

Let S be the circum-centre of the $\triangle ABC$, and A' the mid-pt. of BC . Then SA is \parallel to AO and $= \frac{1}{2}AO$.

Produce SA' to P , making $A'P = SA'$ (or $\frac{1}{2}AO$);

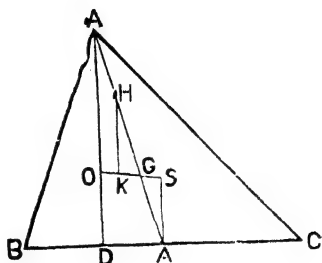


then P is the circum-centre of the $\triangle BOC$. Join SA, PO . Now, SP is equal and \parallel to AO ; \therefore the quadl. SO is a *par*^m. Hence, $SA = PO$. Thus, the circum- \odot s of the \triangle s ABC , BOC , having equal radii, are equal to one another.

Similarly, the circum- \odot s of the \triangle s COA , AOB are each = the circum- \odot of the $\triangle ABC$.

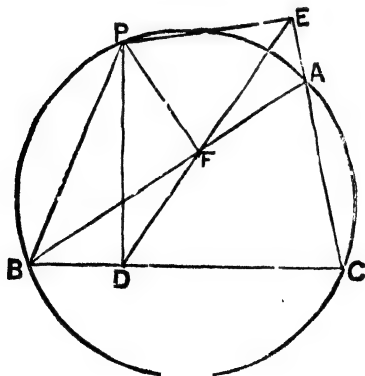
Cor. 2. *The circum-centre, the centroid and the ortho-centre of a triangle are collinear.*

Let S be the circum-centre, and O the ortho-centre, of the $\triangle ABC$; and let A' be the mid-pt. of the side BC . Join SA' and SO ; also join AA' cutting OS in G . Let H, K be the mid-pts. of AG and OG respectively; join HK . Now



HK is \parallel to AO and $= \frac{1}{2} AO$; $\therefore HK$ is equal and \parallel to SA' , and \therefore the quadl. $HKA'S$ is a par^m. (*Bk. I, Th. 22*). But the diagonals of a par^m. bisect each other; $\therefore A'G = GH = \frac{1}{2} AG$. Hence, G is the centroid of the $\triangle ABC$, and it lies on the line SO ; which proves the corollary.

11. *If perpendiculars be drawn to the sides of a triangle from any point on the circum-circle, then the feet of these perpendiculars lie in one straight line.*



From any pt. P on the circum- \odot of the $\triangle ABC$, let PD , PE , PF be drawn \perp s to BC , CA , AB respectively.

Join DF , FE .

To prove that DF , FE are in one str. line.

Proof. Join PA , PB .

Since the $\angle PDB =$ the $\angle PFB$, each being a rt. \angle , the quadl. $PFDB$ is cyclic.

Also, since the \angle s PEA and PFA are supplementary, each being a rt. \angle , the quadl. $PFAE$ is cyclic. (*Th. 15.*)

Now, the quadl. $PBCA$ being cyclic, the $\angle PBC =$ the $\angle PAE$; (*Th.* 14, Cor. 1.*)

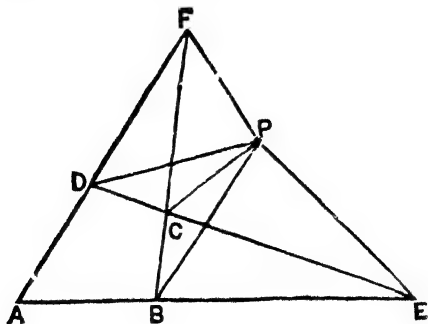
and, the quadl. $PFAE$ being cyclic, the $\angle PAE =$ the $\angle PFE$. (*Th. 11.*)

Hence, the $\angle FBD =$ the $\angle PFE$.

But, the quadl. $PFDB$ being cyclic, the \angle s PBD and PFD are supplementary; \therefore the \angle s PFE and PFD are supplementary, and \therefore DF , FE are in one str. line. Q. E. D.

Note. The line DFE is called the *pedal line* of the point P with respect to the triangle ABC .

12. *The circum-circles of the four triangles formed by four given straight lines, of which no two are parallel, have one point common to them all.*



Let the four str. lines AB, DC, AD, BC form the four Δ s ABF, DCF, ADE, BCE. C is one pt. of intersection of the \odot s DCF, BCF ; let their second pt. of intersection be P.

To prove that P is a pt. on the \odot ABF and also on the \odot ADE.

Proof. Join CP, BP, DP.

The quadl. BCPE being cyclic, the \angle PBE = the \angle PCE ; (Th. 11.)

and the quadl. DCPF being cyclic, the \angle PCE = the \angle PFD. (Th. 14, Cor. 1.)

Hence, the \angle PBE = the \angle PFA,

and \therefore the quadl. ABPF is cyclic. (Th. 15, Cor.)

Thus, P is a pt. on the \odot ABF.

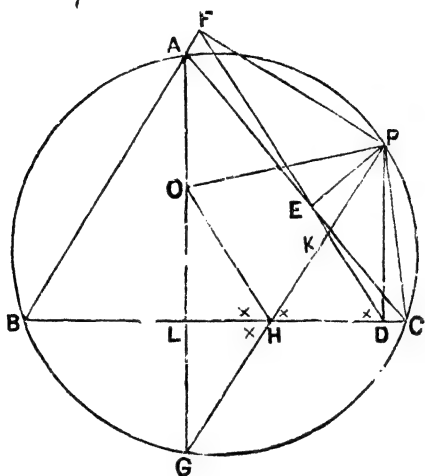
In the same way, it can be shewn that P is a pt. on the \odot ADE.

Note 1. If the four given straight lines are such that the quadl. ABCD is cyclic, then the three pts. F, P, E are collinear.

For, in that case, the \angle FPC = \angle CDA = the \angle CBE ; \therefore the \angle s FPC and EPC are together = the \angle s CBE and EPC = two rt. \angle s. Hence, FP and PE are in the same str. line.

Note 2. If from P the \perp s PH, PK, PL, PM be drawn to the lines ABE, DCE, BCF, ADF respectively, then the four pts. H, K, L, M are collinear. Since P is a pt. on the circum- \odot of the Δ BCE, and PH, PK, PL are the \perp s from P on the sides of a triangle, \therefore the three pts. H, K, L lie in one str. line, (Prop. 12). Similarly, the three pts. K, L, M are in one str. line. Thus, the str. line that passes through K and L also passes through H and M ; which shews that all the four pts. lie in one str. line.

13. *The line joining the ortho-centre O of a triangle ABC to any point P on the circum-circle is bisected by the pedal line of P with respect to the triangle.*



In the above diagram, let DEF be the pedal line of that pt. P .

To prove that OP is bisected by the line DEF .

Proof. Produce AO to meet BC in L and the \odot in G .

Join PG , cutting BC at H , and the pedal line at K .

Join OH , PC . The line AOL is \perp to BC ;

$\therefore OL = LG$. (*Prop. 2, Cor. 2.*)

Hence the \triangle s OLH , GLH are congruent ;
and \therefore the $\angle LHG =$ the $\angle LHO$.

Now, the quadr. PEDC being cyclic,

$$\begin{aligned}
 \text{the } \angle EDH &= \text{the } \angle EPC \\
 &= \text{the complement of the } \angle PCA \\
 &= \text{the complement of the } \angle PGA \\
 &= \text{the } \angle LHG \\
 &= \text{the } \angle LHO;
 \end{aligned}$$

\therefore DF is \parallel to HO.

Again, the $\angle KDH = \text{the } \angle LGH$ (Proved above)
 $= \text{the } \angle KHD$;

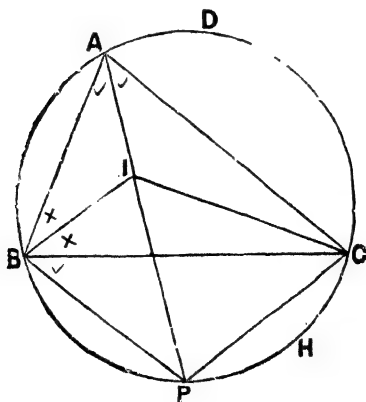
\therefore the $\angle KDP = \text{the } \angle KPD$.

Hence, $KH = KD = KP$.

Thus, K is the mid-pt. of the side PH of the $\triangle OPH$,
 and the line DKF is \parallel to the side HO;

\therefore the side OP is bisected by the line DEF. Q.E.D.

14. *If triangles standing on the same base, on the same side of it, have equal vertical angles, then their in-centres lie on the circumference of a circle whose centre is a known point on the perpendicular bisector of the given base.*



(i) Let $\triangle BAC$ be one of the \triangle s standing on the given base BC and having a given vertical \angle .

Let P be the mid-pt. of the arc BHC of the circum- \odot BAC ; join PA .

Then the $\angle FAB =$ the $\angle PAC$; (*Th. 10, Cor. 2.*)

\therefore the in-centre of the $\triangle BAC$ lies on PA . (*Prob. 5.*)

Let I be the in-centre; join BI, CI .

AB, BC, CA being tangents to the in- \odot , BI, CI are the bisectors of the \angle s B, C respectively. (*Th. 8.*)

Join PB, PC .

Now, the $\angle PBC =$ the $\angle PAC =$ the $\angle IAB$;

\therefore the $\angle PBI =$ the $\angle IAB +$ the $\angle IBA$
 $=$ the $\angle BIP$.

Hence, $PI = PB$; and similarly,
 $PI = PC$.

Thus $PB = PI = PC$ (a)

(ii) Let D be the vertex of *any other* \triangle standing on the base BC , on the same side of it as A , and having the vertical $\angle BDC =$ the $\angle BAC$.

Then the pt. D must be on the arc BAC of the circum- \odot of the $\triangle BAC$. (*Th. 12.*)

Hence, if I' be the in-centre of the $\triangle BDC$, it can be shewn, as before, that $PI' = PB$ (b)

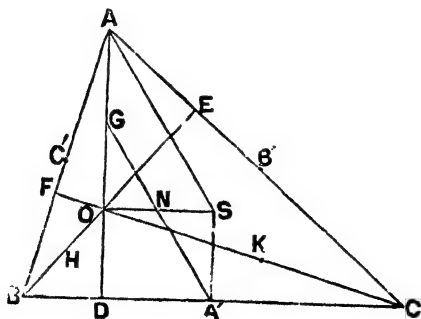
Hence, from (a) and (b), if a \odot be described with P as centre and PB or PC as radius, it will pass through the in-centres of *all* the \triangle s of which the $\triangle BAC$ is one.

Now, since $PB = PC$, the pt. P is evidently on the \perp bisector of BC ;

Thus, P is the pt. where the \perp bisector of BC meets the arc BHC of the common circum- \odot of the \triangle s considered, and is \therefore a *known* pt. Q. E. D.

Note. It should be carefully observed that if the base and the vertical angle of a triangle be given, the triangle admits of various positions. Considering the positions on one side of the base only, it is clear that whatever may be the position, the circum-circle is the same; for, the circle that passes through the extremities of the base and one position of the vertex, also passes through *every other* position of the vertex. Hence, when the base and the vertical angle of a triangle are given, we may say that the circum-circle is *fixed* (i.e. *not different* for different positions of the triangle); and \therefore the circum-centre is *fixed* and the circum-radius is of *constant* magnitude.

15. *In any triangle, the middle points of the sides, the feet of the perpendiculars from the vertices on the opposite sides, and the middle points of the lines joining the ortho-centre to the vertices are concyclic.*



In the above diagram, O is the ortho-centre and S the circum-centre of the $\triangle ABC$; A' , B' , C' are the mid-pt. of

the sides opp. to A, B, C; AD, BE, CF are the \perp s from A, B, C on the opp. sides; and G, H, K are the mid-pts. of the lines OA, OB, OC.

To prove that the nine points A', B', C', D, E, F, G, H, K are concyclic.

Proof. Join SA, SA', SO and GA' cutting SO at N.

Now, SA' is \parallel to AO and $= \frac{1}{2}$ AO;

\therefore SA' is equal and \parallel to GO.

Hence G, O, A', S are the consecutive angular pts. of a parallelogram. (Bk. I, Th. 22.)

Hence GA' and OS bisect each other at N.

Thus GA' is bisected at the mid-pt. of OS; and similarly, HB' and KC' are each bisected at the mid-pt. of OS.

Now, GDA' being a rt. \angle , the \odot described on GA' as diameter passes through D. (Th. 13, Cor. 2.)

\therefore ND = NG = NA', each of them being a radius of this \odot .

Again, since SA' is equal and \parallel to AG,

\therefore SA = A'G. (Bk. I, Th. 22.)

Hence, if R stands for "the circum-radius of the \triangle ABC,"

we have ND = NG = NA' = $\frac{1}{2}$ R. (SA being R.)

Similarly, NE = NH = NB' = $\frac{1}{2}$ R, }

and NF = NK = NC' = $\frac{1}{2}$ R. }

Thus, the nine pts. D, G, A', E, H, B', F, K, C' are each at a distance $= \frac{1}{2}$ R from N, the mid-pt. of OS;

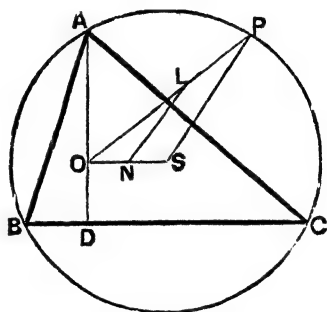
and \therefore they all lie on the \odot^{ce} of the \odot whose centre is the mid-pt. of OG and radius $= \frac{1}{2} R$.

Note. The circle passing through the mid-points of the sides of a triangle is called the **Nine-point circle** of the triangle, from its property of passing through *nine* particular points connected with the triangle. The centre of the nine-point circle may also be conveniently called the "nine-point centre."

Cor. 1. *The nine-point circle of each of the triangles BOC , COA , AOB is the same as that of the triangle ABC .*

Cor 2. *If P is any point on the circum-circle of the triangle ABC , the mid-point of OP is on the nine point circle.*

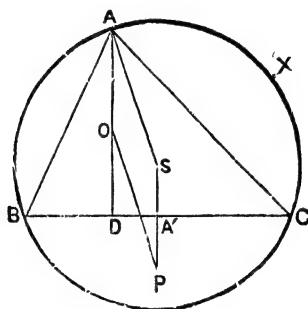
If L is the mid-pt. of OP , join NL . Then NL is \parallel to SP , and $= \frac{1}{2} SP = \frac{1}{2} R$. Thus NL is $=$ a radius of the nine-pt. \odot , and \therefore the pt. L lies on that \odot .



Cor. 3. *Given the base and the vertical angle of a triangle, as the triangle changes its position the centre of the nine-point circle moves along the circumference of a circle whose centre is the mid-point of the given base.*

In the figure of the proposition, suppose XBC to be *any other* position of the triangle, the base BC remaining fixed and the vertical $\angle BXC$ being $=$ the $\angle BAC$. Then X must be on the arc BAC of the circum- \odot of the $\triangle ABC$. Now, if N' be the nine pt. centre of the $\triangle XBC$, we must have $N'A' = \frac{1}{2} SX = \frac{1}{2} SA$. Thus, N and N' are both at the same distance from A' ; which proves the corollary.

16. *Given the base and the vertical angle of a triangle, prove that, as the triangle changes its position, (i) the ortho-centre moves along the circumference of a circle of which the centre is a known point on the perpendicular bisector of the given base, and (ii) the centroid also moves along the circumference of another circle of which the centre is a known point on the same bisector.*



(i) Let ABC be a \triangle standing on the given base BC and having its vertical \angle of the given magnitude.

Let AD be the \perp on BC , O the ortho-centre, S the circum-centre, and A' the mid-pt. of BC .

Join SA' and produce it to P making $A'P = SA'$;

join SA , PO .

Now, SA' is \parallel to AO and $= \frac{1}{2} AO$;

$\therefore SP$ is equal and \parallel to AO .

Hence, $PO = SA$.

(Bk. I. Th. 22.)

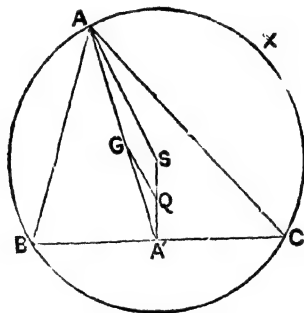
Let XBC be *any other* position of the \triangle , the base BC remaining fixed and the vertical $\angle BXC$ being = the $\angle BAC$.

Then X must be on the arc BAC of the circum- \odot of the $\triangle ABC$.

Now, if O' be the ortho-centre of the $\triangle XBC$, it can be shewn, as before, that $PO' = SX$.

Thus, $PO = PO'$ (each of them being $= SA$), which shews that O and O' are on the \odot^{ce} of a \odot of which the centre is the pt. P and the radius $= SA$.

Also, since $A'P = SA'$, P is obviously a known pt. on the \perp bisector of BC .



(ii) Let ABC be a \triangle standing on the given base BC and having the vertical $\angle BAC$ of the given magnitude.

Let A' be the mid-pt. of BC , and G the centroid ; then G is a pt. on AA' such that $AA' = 3GA'$.

Let S be the circum-centre ; join SA' , SA .

Draw $GQ \parallel$ to AS , meeting SA' in Q .

Now, since AS is to GQ and $AA' = 3A'G$,

$$\therefore AS = 3GQ. \quad (\text{Bk. I. Sec. VIII, Prop. 5.})$$

$$\therefore GQ = \frac{1}{3}AS.$$

Let XBC be *any other* position of the \triangle , the base BC remaining fixed and the vertical $\angle BXC$ being = the $\angle BAC$.

The X must be on the arc ABC of the circum- \odot of the $\triangle ABC$.

Now, if G' be the centroid of the $\triangle XBC$, it can be shewn, as before, that $G'Q = \frac{1}{3}XS$.

Thus, $GQ = G'Q$ (each of them being $= \frac{1}{3}SA$), which shews that G and G' are on the \odot^c of a \odot of which the centre is the pt. Q, and the radius $= \frac{1}{3}SA$.

Also, since $QA' = \frac{1}{3}SA'$, Q is obviously a known pt. on the \perp bisector of BC.

EXERCISE (27).

1. Shew that the middle points of the chords of a circle which pass through a given point all lie on the circumference of another circle.

2. If a quadrilateral be circumscribed about a given circle, prove that the sum of any pair of opposite sides is equal to the sum of the other pair. Hence prove that any parallelogram circumscribed about a circle is a rhombus.

3. Prove that the triangle formed by joining the three points in which the in-circle of a triangle meets the sides, is acute-angled.

4. The diagonals AC, BD of a parallelogram ABCD intersect at O. Shew that the circum-circles of the triangles AOB, COD touch each other at O.

5. Shew that the greatest quadrilateral that can be inscribed in a circle is a square.

6. On the sides BC, CA, AB of a given triangle ABC, any three points D, E, F respectively are taken. Prove that the circum-circles of the triangles EAF, FBD, DCE meet in a point.

7. If the in-circle of a triangle ABC touches the sides BC, CA, AB at D, E, F respectively, prove that $BC + AF = CA + BD = AB + CE =$ the semi-perimeter of the triangle ABC.

8. With the vertices A, B, C of a triangle as centres construct three circles each of which shall touch the other two.

9. Prove that the sum of any three alternate angles of a hexagon inscribed in a circle is equal to four right angles.

10. AB is the common chord of two circles, and P is any point on the circumference of one of them so that the straight lines PAQ , PBR meet the circumference of the other in Q and R . Prove that the arc QR is of the same magnitude for different positions of P .

11. The opposite sides of a cyclic quadrilateral are produced to meet in P and Q , and about the triangles so formed without the quadrilateral circles are described, which meet in R . Prove that the points P , R , Q are collinear.

12. ABC is a triangle; BCA' , CAB' , ABC' are equilateral triangles so that the points (A, A') , (B, B') , (C, C') are on opposite sides of BC , CA , AB respectively. Prove that the circum-circles of the three equilateral triangles meet at a point which is also the point of concurrence of the lines AA' , BB' , CC' .

13. If an equilateral triangle be inscribed in a circle, and the adjacent arcs cut off by two of its sides be bisected, prove that the line joining the points of bisection is trisected by the sides,

14. P is any point on the circum-circle of a given triangle ABC ; PA' , PB' , PC' are chords of the circle perpendicular respectively to BC , CA , AB . Shew that the triangles ABC , $A'B'C'$ are congruent.

15. If S be the circum-centre of a triangle ABC , and if the perpendicular from S on BC meets the circum-circle in K and L , L being on the same side of BC as A , prove that AK and AL bisect the interior and exterior angles at A . Also, if I be the in-centre, prove that K is the circum-centre of the triangle BIC .

16. ABC is a triangle; O is the point of intersection of the perpendiculars AD , BE , CF upon the sides of the triangle; G , H , K are the middle points of the sides, and L , M , N the middle points of the lines OA , OB , OC . Prove that each of the angles LHG , LKG is a right angle; and hence prove that the circle passing through the points G , H , K also passes through the six points (L, D) , (M, E) , (N, F) .

17. Given one angle, of a triangle, the side opposite to it, and the sum of the other two sides, construct the triangle.

18. ABC is any triangle, inscribed in a circle, and AP , EQ are chords of the circle parallel to BC , CA respectively. Prove that PQ is parallel to the tangent at C .

19. AB is the diameter of a semi-circle, D and E are any two points on its circumference. AE , BD intersect at M ; and AD , BE produced intersect at L . Prove that LM produced cuts AB at right angles.

20. BC is a given arc of a circle whose centre is O ; A is any point in BC . AD , AE are drawn perpendiculars to OB , OC . Prove that the line DE is of constant length.

21. Prove that an equiangular polygon inscribed in a circle has its alternate sides equal. Hence, shew that an equiangular pentagon inscribed in a circle is also equilateral.

22. If any number of triangles be on the same base, on the same side of it, and have equal vertical angles, prove that the bisectors of the vertical angles are concurrent.

23. If two circles intersect and if one of them passes through the centre of other, prove that the tangents to the latter at the points of intersection will meet on the former.

24. If P be any point on the circle circumscribed about a given equilateral triangle ABC , prove that one of the lines PA , PB , PC is equal to the sum of the other two.

25. O is the circum-centre of a triangle ABC , and D , E , F are the feet of the perpendiculars from A , B , C on the opposite sides. Shew that OA , OB , OC are respectively perpendiculars to EF , FD , DE .

26. Right-angled triangles are described on the same hypotenuse. Shew that the locus of the centres of the in-circles is a quarter of the circumference of a circle of which the common hypotenuse is a chord.

27. Construct a triangle, having given one side and the radii of the in-circle and circum-circle.

28. $ABCDE$ is a regular pentagon inscribed in a circle, and AC , BD intersect at O . Prove that $AO = DO$, and that $BC^2 = AC \cdot CO$.

29. $ACDB$ is a semi-circle, AB being the diameter, and the two chords AD , BC intersect at E . If a circle be circumscribed about the triangle CDE , prove that it will cut the former at right angles.

30. If I be the in-centre, and I_1 , I_2 , I_3 , the ex-centres of a given triangle, prove that I is the ortho-centre of the triangle $I_1I_2I_3$. Hence shew that the circum-circles of the four triangles II_2I_3 , II_3I_1 , II_1I_2 and $I_1I_2I_3$ are equal to one another.

31. The diagonals of a given quadrilateral $ABCD$ intersect at O . Shew that the centres of the circles circumscribed about the triangles OAB , OBC , OCD , ODA are at the angular points of a parallelogram.

32. If a line BC of constant length have its extremities on two fixed str. lines AX , AY , prove that the circum-radius of the $\triangle ABC$ is of the same length for all positions of BC . Hence prove that, as BC changes its position, the circum-centre of the $\triangle ABC$ moves on the \odot^{ve} of a \odot .

33. In the preceding example, shew that the distance of the circum-centre from the line BC is the same for all positions of BC . Hence prove that, as BC changes its position, the ortho-centre of the triangle ABC moves on the circumference of a circle.

34. ABC is an equilateral triangle inscribed in a circle whose centre is O , and BO is produced to meet the circumference in D . Prove that the arc AD is one-sixth of the whole circumference. Hence prove that $AD = AO$, and that $AB^2 = 3OA^2$.

35. PM , PN are tangents to a circle, of which the centre is O , and MN the chord of contact. PEF is any straight line cutting the circle at E , F . OS is drawn perpendicular to EF and produced to meet MN produced in Q . Prove that $OS.OQ = (\text{radius})^2$, and hence prove that QE , QF are tangents to the circle.

36. Through a given point without a given circle draw a straight line so that the part intercepted by the circumference may be equal to a given straight line not greater than the diameter.

37. The internal and external bisectors of the angle A of a triangle meet the base BC in E, E' and the circum-circle in D and D' . Prove that D is the ortho-centre of the triangle $EE'D'$.

38. From S , the circum-centre of the triangle ABC , perpendiculars SA', SB', SC' are drawn to the sides, and these perpendiculars are produced to P, Q, R respectively so that $SA' = A'P, SB' = B'Q$ and $SC' = C'R$. Prove that S is the ortho-centre of the triangle PQR , and hence shew that the $\triangle s$ ABC, PQR have the same nine-point circle.

39. Construct a triangle, having given the ortho-centre, the circum-centre and one angular point.

40. AB and AC are two given str. lines in which B and C are two given pts. BD is drawn \perp to AC , and $DE \perp$ to AB ; in like manner CF is drawn \perp to AB , and FG to AC . Prove that EG is \parallel to BC .

41. If the centres of two circles which touch each other externally be fixed, prove that the external common tangents of the two circles will also touch the circle of which the straight line joining the fixed centres is the diameter.

42. AB, AC are the equal sides of an isosceles triangle ABC . P is a point on AB , and Q a point on AC produced such that $BP = CQ$. Prove that, for all positions of PQ , the circum-circle of the triangle APQ passes through a fixed point on the bisector of the angle BAC .

43. ABC is a triangle, and BE, CF are the perpendiculars from B and C upon the opposite sides. If K be the mid-pt. of BC , prove that $KF = KE$, and that each of the $\angle s$ KFE, KEF is equal to the $\angle A$.

44. If two circles touch each other internally, prove that any chord of the greater circle which touches the smaller is divided at the point of contact into segments which subtend equal angles at the point of contact of the two circles.

45. PA, PB, PC are any three chords of a circle. Prove that the circles described on these chords as diameters will intersect again in three points which are collinear.

46. A series of circles touch a fixed straight line at a fixed point. Shew that the tangents at the points where they cut a parallel fixed straight line all touch a fixed circle.

47. AD, BE, CF are the perpendiculars of the triangle ABC. Prove that the feet of the \perp s from D on AB, AC, BE, CF are collinear.

48. O is the ortho-centre of a $\triangle ABC$, and P, Q, R the circum-centres of the \triangle s BOC, COA, AOB respectively. Prove that O is the circum-centre of the $\triangle PQR$. Prove also that each of the quadls. OPCQ, OQAR and ORBP is a rhombus, and that the \triangle s ABC, PQR are congruent.

49. Having given the base and the vertical angle of a triangle, prove that the nine-point circle touches a fixed circle whose radius is equal to that of the circum-circle.

50. A man standing on the deck of a ship in the middle of a calm ocean looks around with a telescope. Shew that the distance from him of the farthest object he can see on the surface of the ocean is approximately $\sqrt{\frac{1}{3}}$ mile, where 1 inch is the height of the observer's eye above the water, assuming the radius of the earth to be 4000 miles.

BOOK IV.

Proportion ; Similar Triangles.

SECTION I.

FUNDAMENTAL IDEAS AND DEFINITIONS.

1. A straight line, of an arbitrarily chosen length, which is used for the purpose of ascertaining the lengths of other straight lines, is called the **unit of length**. We are said to **measure** a straight line when we ascertain the number of times that it contains the unit of length.

2. The area of a square of which a side is equal to the unit of length is called the **unit of area**. We are said to **measure** the area of a plane figure when we ascertain the number of times that this area contains the unit of area.

3. An angle, of an arbitrarily chosen size, which is used for the purpose of ascertaining the sizes of other angles, is called the **unit of angle**. We are said to **measure** an angle when we ascertain the number of times that it contains the unit of angle.

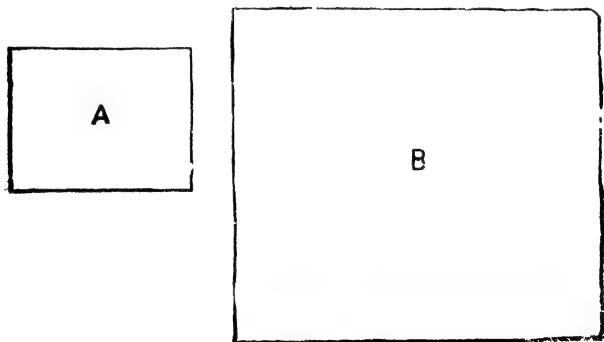
4. Any thing that can be measured is called a **magnitude**. Clearly therefore, a straight line, an angle and the area of a plane figure are all *magnitudes*.

Note. This is the more extended sense in which the word "magnitude" is used. For other meanings of the term see Art. 5, Sec. 1, Book I.

5. The **numerical measure** of any given magnitude is the number of times that it contains the unit of its kind. Thus, if one inch be taken as the unit of length, the *numerical measure* of a straight line that contains six complete inches with a remainder which is two-thirds of an inch long, is $6\frac{2}{3}$.

Note. The number of times that any given magnitude contains the unit of its kind, is said to **represent** that magnitude. Thus, if one foot be taken as the unit of length, a straight line which is 9 inches long will be *represented* by the number $\frac{3}{4}$.

6. If two magnitudes A and B be such that B can be divided into parts each of which is equal to A, then A is said to be a **measure** of B, and B is said to be a **multiple** of A. Thus, in the following diagram, if the areas of the rectangles A and B be $\frac{3}{4}$ of a square inch and $3\frac{3}{4}$

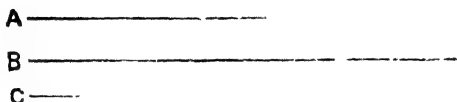


square inches respectively, then it is easy to see that B can be divided into five parts each of which is equal in area to A; hence, A is a *measure* of B, and B is a *multiple* of A.

Note 1. In the above illustration, the expression "**A** is a measure of **B**" means *the area of A is a measure of the area of B*. Similar to the expression "**B** is a multiple of **A**" means *the area of B is a multiple of the area of A*.

Note 2. When **B** is divisible into parts each of which is equal to **A**, it may be said that **B** contains **A** an integral number of times.

Note 3. If two magnitudes have a common measure, they have also a common multiple. Let **A**, **B** and **C** be three straight lines such that **C** is a common measure of **A** and **B**.



Then, we must have

$$\text{and} \quad \left. \begin{array}{l} A = m \text{ times } C \\ B = n \text{ times } C \end{array} \right\} \text{ where } m \text{ and } n \text{ are two integers.}$$

$$\text{Hence, } \left. \begin{array}{l} nA = mn.C \\ \text{and } mB = mn.C \end{array} \right\}$$

$$\text{and } \therefore nA = mB.$$

Thus, the straight line which is n times **A** is also that which is m times **B**, and is therefore a common multiple of **A** and **B**.

7. If two magnitudes of the same kind have a common measure they are said to be **commensurable**; if they have no common measure they are said to be **incommensurable**.

Note. The side and the diagonal of a square are a pair of **incommensurable** magnitudes. For, if the length of the side be a inches, the length of the diagonal must be $a\sqrt{2}$ inches i.e., ($a \times 1.4142135 \dots$) inches.

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Hence, the diagonal = a inches $\times 1\cdot4142135\ldots$

$$= \left(\frac{a \text{ inches}}{100} \right) \times 141\cdot42135\ldots ;$$

which shews that one-hundredth part of the side, which of course is a measure of the side, is not contained an integral number of times in the diagonal, and is therefore *not* a measure of the diagonal.

$$\text{Again, the diagonal} = \left(\frac{a \text{ inches}}{215} \right) \times (215 \times 1\cdot414\ldots)$$

$$= \left(\frac{a \text{ inches}}{215} \right) \times 304\cdot011\ldots ;$$

which shews that one 215th part of the side, which of course is a measure of the side, is *not* a measure of the diagonal.

And so on. Clearly therefore *no measure* of the side is a measure of the diagonal, and consequently the side and the diagonal *have no common measure*.

8. If two magnitudes are incommensurable they may still be regarded as commensurable for all *practical purposes*. For instance, take a square of which the side is ten inches in length ; then

$$\begin{aligned} \text{(i) The diagonal} &= (10 \text{ inches}) \times 1\cdot414213562\ldots \\ &= \left[\frac{10 \text{ inches}}{1,00,000} \right] \times 141421\cdot3562\ldots \\ &= (\text{one ten-thousandth of an inch}) \\ &\quad \times 141421\cdot3562\ldots \\ &= 141421 \text{ times (one ten-thousandth} \\ &\quad \text{of an inch)} + \cdot3562\ldots \text{ of (one} \\ &\quad \text{ten-thousandth of an inch)} ; \end{aligned}$$

which shews that if the portion of the diagonal which is 141421 times (one ten-thousandth of an inch) be taken as the diagonal, leaving out the remainder which is *less than one ten-thousandth of an inch*, then the side and the diagonal may be regarded as *commensurable*, the common measure being one ten-thousandth of an inch.

$$\begin{aligned}
 \text{(ii) The diagonal} &= \left(\frac{10 \text{ inches}}{10,00,000} \right) \times 1414213 \cdot 562 \dots\dots \\
 &= (\text{one hundred-thousandth of an inch}) \\
 &\quad \times 1414213 \cdot 562 \dots\dots \\
 &= 1414213 \text{ times (one hundred-thou-} \\
 &\quad \text{sandth of an inch)} + \cdot 562 \dots\dots \text{of} \\
 &\quad (\text{one hundred-thousandth of an inch}) ;
 \end{aligned}$$

which shews that if the portion of the diagonal which is 1414213 times (one hundred-thousandth of an inch) be taken as the diagonal, leaving out the remainder which is *less than one hundred-thousandth of an inch*, then the side and the diagonal may be regarded as *commensurable*, the common measure being one hundred-thousandth of an inch.

$$\begin{aligned}
 \text{(iii) The diagonal} &= \left[\frac{10 \text{ inches}}{10,000,000} \right] \times 14142135 \cdot 62 \dots\dots\dots \\
 &= (\text{one millionth of an inch}) \\
 &\quad \times 14142135 \cdot 62 \dots\dots\dots \\
 &= 14142135 \text{ times one millionth of} \\
 &\quad \text{an inch} + \cdot 62 \dots\dots \text{of one millionth of} \\
 &\quad \text{an inch} ;
 \end{aligned}$$

which shews that if the portion of the diagonal which is 14142135 times one-millionth of an inch be taken as the diagonal, leaving out the remainder which is *less than one-millionth of an inch*, then the side and the diagonal may be regarded as *commensurable*, the common measure being one-millionth of an inch.

And so on

From the above it is clear that if two magnitudes are incommensurable they may still be regarded as commensurable, the common measure being a *very small* measure of one of them ; and the smaller this measure is, the smaller is the consequent error. In other words, if two magnitudes are incommensurable, one of them may be regarded as a *very large* multiple of a *very small* measure of the other, the consequent error being too small for *practical* purposes.

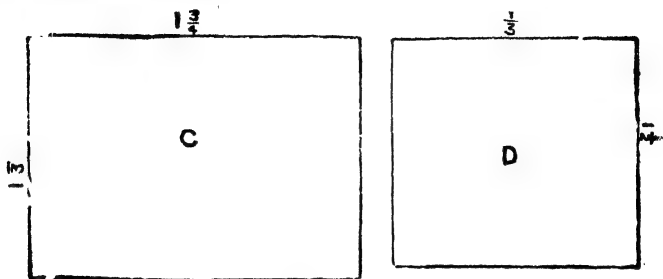
9. The ratio of one magnitude to another of the same kind is the number which expresses what multiple or what fraction the former is of the latter.

(i) If A and B are two straight lines whose lengths are 3 inches and 1 inch respectively.



then the ratio of A to B is 3 ; and the ratio of B to A is $\frac{1}{3}$.

(ii) If C and D are two rectangles whose areas are $2\frac{1}{3}$ and $1\frac{2}{3}$ square inches respectively,



then the ratio of C to D is $\frac{2\frac{1}{3}}{1\frac{2}{3}}$, which is $=\frac{7}{5}$; and the ratio of D to C is $\frac{1\frac{2}{3}}{2\frac{1}{3}}$, which is $=\frac{5}{7}$.

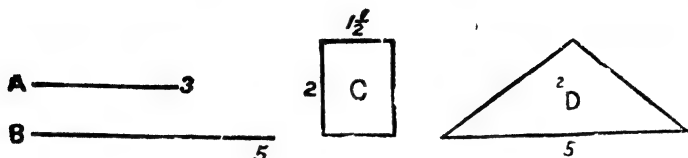
(iii) Generally speaking, if A and B are two magnitudes of the same kind, whose *numerical measures* referred to one and the same unit are a and b respectively, then the ratio of A to B is $\frac{a}{b}$ and the ratio of B to A is $\frac{b}{a}$.

Note 1. The symbol $\frac{A}{B}$ stands for "the ratio of A to B." Hence, "the ratio of A to B is $\frac{a}{b}$," is briefly written as $\frac{A}{B} = \frac{a}{b}$, "The ratio of A to B" is also written as $A : B$.

Note 2. "The ratio of A to B is $\frac{a}{b}$ " and "A and B are in the ratio of a to b " are equivalent expressions.

Note 3. When we consider the ratio of A to B, A is called the **antecedent** (or the first term) and B the **consequent** (or the second term) of the ratio. When the ratio of B to A is considered, B is the **antecedent** and A the **consequent**.

Note 4. If **A** and **B** are two straight lines whose numerical measures referred to one and the same unit of length are 3 and 5 respectively; and if **C** and **D** are two areas of which the numerical measures referred to one and the same unit of area are respectively



3 and 5; then the ratio of **A** to **B** is $\frac{3}{5}$, and also the ratio of **C** to **D** is $\frac{2}{5}$. Hence, *the ratio of A to B is equal to that of C to D*. This is briefly expressed by the notation, $\frac{A}{B} = \frac{C}{D}$.

10. Four magnitudes **A**, **B**, **C**, **D** are said to be **proportional** (or **in proportion**) when $\frac{A}{B} = \frac{C}{D}$, i.e.,

when the ratio of **A** to **B** is equal to that of **C** to **D**.

Note 1. (i) $\frac{A}{B} = \frac{C}{D}$.

(ii) **A** : **B** = **C** : **D**, and

(iii) **A** : **B** :: **C** : **D** are equivalent expressions and read as "**A** is to **B** as **C** is to **D**."

Note 2. Each of the above three expressions is called a **proportion**.

Note 3. In a proportion such as **A** : **B** = **C** : **D**, **A** and **D** are called the **extremes**, and **B** and **C** the **means**.

Note 4. The word "proportional" is also used as a noun; thus, in the proportion **A** : **B** :: **C** : **D**, **D** is said to be the **fourth proportional** to **A**, **B** and **C**.

Note 5. In a proportion, terms occupying similar positions, i.e., terms which are both antecedents or both consequents of the ratios, are said to be **corresponding** or **homologous terms**.

11. Magnitudes A, B, C, D of the same kind are said to be in **continued proportion** when $\frac{A}{B} = \frac{B}{C} = \frac{C}{D}$, *i.e.*, when the ratio of A to B, the ratio of B to C, the ratio of C to D, are all equal

12. When three magnitudes A, B, C of the same kind are in continued proportion, *i.e.*, when

$$\frac{A}{B} = \frac{B}{C},$$

B is said to be the **mean proportional** to A and C; and C is said to be the **third proportional** to A and B.

13. If three magnitudes A, B, C of the same kind be such that the ratio of B to A is equal to that of C to A, then $B = C$. For, let a, b, c be the numerical measures of A, B, C referred to one and the same unit; then, by hypothesis,

$$\frac{b}{a} = \frac{c}{a},$$

and $\therefore b = c$. Hence $B = C$.

Thus, **magnitudes which have the same ratio to the same magnitude are equal to one another.**

Note. Conversely, equal magnitudes have the same ratio to the same magnitude.

14. If four magnitudes A, B, C, D be such that the ratio of A to B is equal to that of C to D, then the ratio of B to A is also equal to that of D to C. For, if a, b be the numerical measures of A and B referred to a unit of the class to which A and B belong, and if c, d be the numerical

measures of C and D referred to a unit of the class to which C and D may belong; then, by hypothesis,

$$\frac{a}{b} = \frac{c}{d} \quad \bullet$$

$$\therefore 1 \div \frac{a}{b} = 1 \div \frac{c}{d}; \quad \bullet$$

$$\text{or,} \quad \frac{b}{a} = \frac{d}{c}; \text{ whence } \frac{B}{A} = \frac{D}{C}.$$

Thus, if four magnitudes be proportional, they are also proportional when taken inversely.

15. If four magnitudes A, B, C, D of the same kind be such that the ratio of A to B is equal to that of C to D, then also is the ratio of A to C equal to that of B to D. For, if a, b, c, d be the numerical measures of A, B, C, D respectively, referred to one and the same unit, then, by hypothesis,

$$\frac{a}{b} = \frac{c}{d};$$

$$\therefore \frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c};$$

$$\text{or,} \quad \frac{a}{c} = \frac{b}{d}; \text{ whence } \frac{A}{C} = \frac{B}{D}.$$

Thus, if four magnitudes of the same kind be proportional, they are also proportional when taken alternately.

16. If four magnitudes A, B, C, D be such that the ratio of A to B is equal to that of C to D, then also is the ratio of A + B to B equal to that of C + D to D. For, if

a , b be the numerical measures of A and B referred to a unit of the class to which A and B belong, and if c , d be the numerical measures of C and D referred to a unit of the class to which C and D may belong, then, by hypothesis,

$$\frac{a}{b} = \frac{c}{d};$$

$$\therefore \frac{a}{b} + 1 = \frac{c}{d} + 1;$$

$$\text{or, } \frac{a+b}{b} = \frac{c+d}{d}; \text{ whence } \frac{A+B}{B} = \frac{C+D}{D}.$$

Thus, if four magnitudes be proportional, the sum of the first and second is to the second as the sum of the third and fourth is to the fourth. This inference is spoken of as **componendo**.

Note 1. If A be greater than B , and C greater than D , it is easy to prove that $\frac{A-B}{B} = \frac{C-D}{D}$.

This inference is spoken of as **Dividendo**.

$$\begin{array}{l} \text{Note 2. Since } \left. \begin{array}{l} \frac{a+b}{b} = \frac{c+d}{d} \\ \text{and } \frac{a-b}{b} = \frac{c-d}{d} \end{array} \right\} \therefore \begin{array}{l} \frac{a+b}{a-b} = \frac{c+d}{c-d} \\ \text{whence } \frac{A+B}{A-B} = \frac{C+D}{C-D} \end{array} \end{array}$$

This inference is spoken of as **Componendo and Dividendo**.

17. If the ratios of A to B , C to D , E to F be equal to one another, where A, B, C, D, E, F are all magnitudes of the same kind, then each of these ratios is equal to the ratio of $A+C+E$ to $B+D+F$. For, if a, b, c, d, e, f be

the numerical measures of A, B, C, D, E, F respectively, referred to one and the same unit, then, by hypothesis,

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f}.$$

Taking each of these fractions = k , we have

$$a = bk, c = dk, e = fk;$$

$$\therefore a + c + e = k(b + d + f);$$

$$\therefore k = \frac{a + c + e}{b + d + f},$$

$$\text{i.e., } \frac{a}{b} \text{ or } \frac{c}{d} \text{ or } \frac{e}{f} = \frac{a + c + e}{b + d + f}.$$

$$\text{Hence, } \frac{A}{B} \text{ or } \frac{C}{D} \text{ or } \frac{E}{F} = \frac{A + C + E}{B + D + F}.$$

Thus, if any number of ratios be equal, where the antecedents and consequents are all magnitudes of the same kind, then any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.

This inference is spoken of as **Addendo**.

18. $A : B :: C : D$, where A, B, C, D are four straight lines, then the rectangle contained by A and D is equal to the rectangle contained by B and C. For, if a, b, c, d be the numerical measures of A, B, C, D respectively, referred to one and the same unit, then, by hypothesis,

$$\frac{a}{b} = \frac{c}{d};$$

$$\therefore \frac{a}{b} \times bd = \frac{c}{d} \times bd,$$

$$\text{or, } ad = bc.$$

But ad and bc are the numerical measures of the areas of the rectangles $A.D$ and $B.C$ respectively.

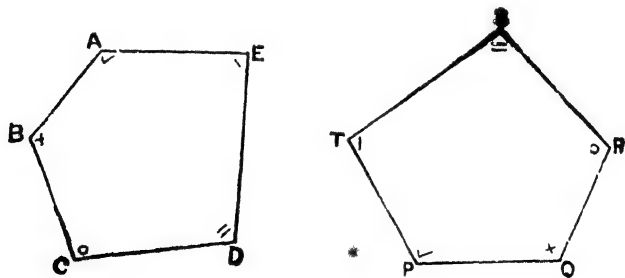
Hence the rectangle contained by A and D = the rectangle contained by B and C .

Thus, if four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means.

Conversely, if four straight lines A, B, C, D are such that $A.D = B.C$, then $A : B = C : D$.

Note. It follows from the above that when three straight lines are in continued proportion the rectangle contained by the extremes is equal to the square on the mean. Conversely, if three straight lines A, B, C are such that $AC = B^2$, then $A : B = B : C$.

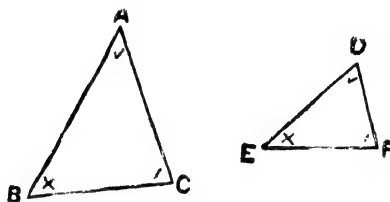
19. Two rectilinear figures are said to be *equiangular* when the angles of the one, taken in order, are respectively equal to those of the other, taken in order.



In the preceding diagram, the pentagons $ABCDE$ and $PQRST$ are *equiangular*, the \angle s A, B, C, D, E being respectively equal to the \angle s P, Q, R, S, T .

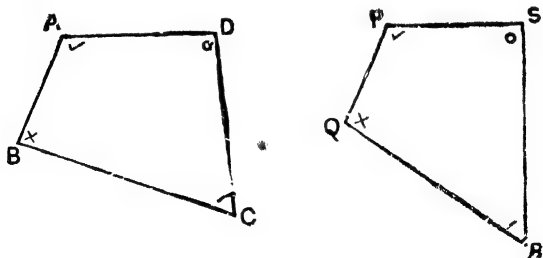
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20. In two equiangular triangles, any angle of the first and that angle of the second to which it is equal are said to be **corresponding angles**; and sides *opposite* to corresponding angles are said to be **corresponding sides**. Thus, if in the $\triangle s$ ABC, DEF, the $\angle s$ A, B, C are respectively equal to the $\angle s$ D, E, F, then



the $\angle s$ A and D are a pair of *corresponding angles*; and so are the $\angle s$ B and E, as well as the $\angle s$ C and F. Also the sides BC and EF are a pair of *corresponding sides*; and so are the sides CA and FD, as well as the sides AB and DE.

Note. Generally speaking, in two *equiangular* rectilineal figures, any angle of the first and that angle of the second to which it is equal.



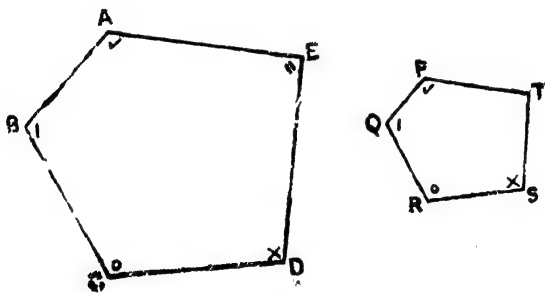
are said to be *corresponding angles*; also the common arm of any two consecutive angles of the first and the common arm of those two

consecutive angles of the second to which they are respectively equal are said to be **corresponding sides**. Thus, if in the preceding two figures **ABCD** and **PQRS**, the \angle s **A, B, C, D** be respectively equal to the \angle s **P, Q, R, S** then the \angle s **A** and **P** are a pair of corresponding angles; so are the \angle s **B** and **Q**, the \angle s **C** and **R**, and the \angle s **D** and **S**. Again the \angle s **B** and **C** being respectively equal to the \angle s **Q** and **R**, the side **BC** (the common arm of the \angle s **B** and **C**) and the side **QR** (the common arm of the \angle s **Q** and **R**) are a pair of *corresponding sides*; so are **CD** and **RS**, **DA** and **SP**, **AB** and **PQ**.

[Of two *corresponding angles*, or two *corresponding sides*, either is said to *correspond* to the other.]

21. If two rectilineal figures are equiangular and if the ratio of any side of the first to the corresponding side of the second is the same, then they are said to be **similar**.

Thus, in the following diagram, the figures **ABCDE** and **PQRST** are *similar*,

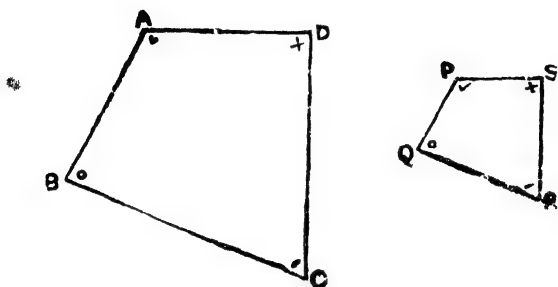


(i) if the \angle s **A, B, C, D, E** are respectively equal to the \angle s **P, Q, R, S, T**,

and (ii) if $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CD}{RS} = \frac{DE}{ST} = \frac{EA}{TP}$.

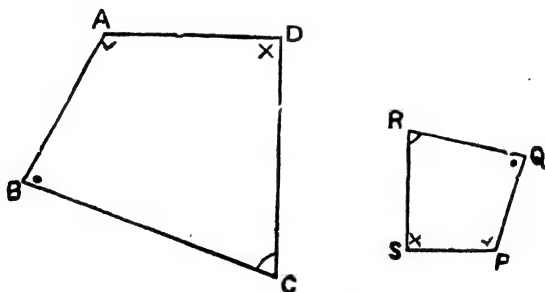
22. Two similar rectilineal figures are said to be **similarly situated** when their corresponding sides are parallel and the same in *sense*. Thus, in diagram (1), the two figures are

(1)



similar as well as *similarly situated*; whilst in diagram (2)

(2)

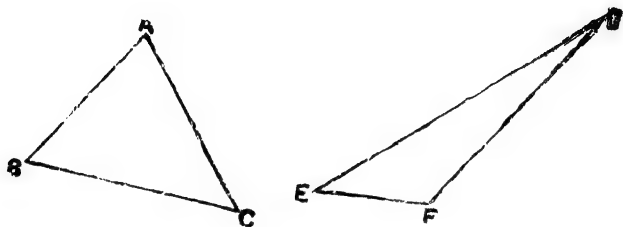


the two figures are similar but *not* similarly situated the cause the corresponding sides, though parallel, are opposite in *sense*.

23. Two rectilineal figures are said to have their sides about two of their angles **reciprocally proportional** when a side of the first is to a side of the second as the remaining side of the second is to the remaining

side of the first. Thus, the \triangle s ABC and DEF, have their sides about the \angle s B and E *reciprocally proportional*,

$$\text{if } \frac{AB}{DE} = \frac{EF}{BC}.$$



24. If four str. lines A, B, C, D be such that

$$\frac{A}{B} = \frac{C}{D}, \text{ then also}$$

$$\frac{\text{the sq. on } A}{\text{the sq. on } B} = \frac{\text{the sq. on } C}{\text{the sq. on } D}.$$

A —————
B —————

C —————
D —————

Let m, n be the numerical measures of A and B referred to some common unit ; then m and n are also the numerical measures of C and D referred to some *other* unit. The units of length being different, the units of area are also different in the two cases.

If α denote the unit of area in the first case and β , that in the second, we must have

$$\left. \begin{array}{l} \text{Sq. on A} = m^2\alpha \\ \text{sq. on B} = n^2\alpha \end{array} \right\} \qquad \left. \begin{array}{l} \text{sq. on C} = m^2\beta \\ \text{sq. on D} = n^2\beta \end{array} \right\}.$$

$$\text{Hence } \frac{\text{sq. on A}}{\text{sq. on B}} = \frac{\text{sq. on C}}{\text{sq. on D}}, \quad \left(\text{each of these two ratios} \right. \\ \left. \text{being} = \frac{m^2}{n^2} \right).$$

EXERCISE (28).

1. Define the *unit of length* and *unit of area*. If 3 inches be the unit of length, what is the unit of area? Into how many units of area can a rectangle be divided whose sides are 9 and 15 inches respectively? Illustrate your answer by a diagram.

2. When is one magnitude said to be a *multiple* of another? and when a *measure*? If two magnitudes have a common multiple, prove that they have also a common measure.

3. When are two magnitudes said to be *commensurable*? If A and B denote any two magnitudes that are commensurable, prove that, A being divided into a certain number of equal parts, any of these parts is contained an exact number of times in B.

4. When are two magnitudes of the same kind said to be *incommensurable*? If the length of the diagonal of a square be d inches, show that a side of the square = $(d \times .70710678\dots)$ inches. Hence prove that the diagonal and the side have no common measure.

5. In the preceding example, shew that the diagonal and the side may be regarded as having a common measure, if a very small error be neglected.

6. What is meant by the *ratio* of one magnitude to another of the same kind? Give illustrations.

7. What is the meaning of $\frac{A}{B} = \frac{C}{D}$?

For the purpose of this relation, is it necessary that A, B, C, D should all be magnitudes of the same kind? If not, how may the matter stand?

8. If three magnitudes A, B, C of the same kind?

such that $\frac{B}{A} = \frac{C}{A}$, prove that $B = C$.

9. If four magnitudes A, B, C, D be such that $\frac{A}{B} = \frac{C}{D}$,

prove that $\frac{B}{A} = \frac{D}{C}$.

10. If four magnitudes A, B, C, D of the same kind.

be such that $\frac{A}{B} = \frac{C}{D}$, prove that $\frac{A}{C} = \frac{B}{D}$.

11. If A, B, C, D be four straight lines such that

$\frac{A}{B} = \frac{C}{D}$, prove that the rect. A, D = the rect. B, C.

12. If A, B, C be three given straight lines such that

$A.C = B^2$, prove that $\frac{A}{B} = \frac{B}{C}$.

13. When are two rectilinear figures said to be *equiangular*? Illustrate your meaning by a diagram.

14. What are *corresponding angles* and *corresponding sides* in two equiangular rectilinear figures? Illustrate your meaning by a diagram.

15. When are two rectilinear figures said to be *similar*? Illustrate your meaning by a diagram.

16. When are two similar rectilinear figures said to be *similarly situated*? Illustrate your meaning by a diagram.

17. If four straight lines A, B, C, D be such that

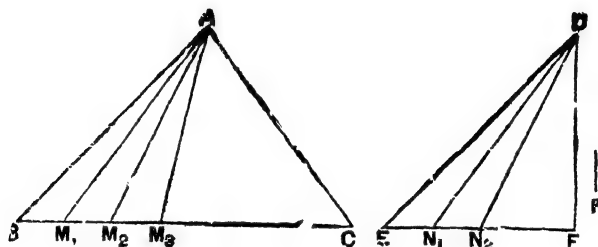
$\frac{A}{B} = \frac{C}{D}$, prove that $\frac{A^2}{B^2} = \frac{C^2}{D^2}$.

SECTION II.

THEOREMS.

Theorem 1. (EUC. VI. 1.)

If two triangles have equal altitudes their areas are to one another as their bases.



Let the \triangle s ABC, DEF, standing on bases BC and EF respectively, have equal altitudes.

To prove that the $\triangle ABC$: the $\triangle DEF$
 $= BC : EF$.

Proof. Let the str. line P be a common measure of BC and EF ; and suppose that P is contained m times in BC and n times in EF. Then $\frac{BC}{EF} = \frac{m}{n}$ (1)

From BC cut off BM_1, M_1M_2, M_2M_3 , &c., each = P ; and from EF cut off EN_1, N_1N_2 , &c., each = P. Then BC is divided into m equal parts at M_1, M_2, M_3 , &c., and EF is divided into n equal parts at N_1, N_2 , &c.

Join $AM_1, AM_2, AM_3, \&c., DN_1, DN_2, \&c.$

Now, it is clear that $BM_1 = M_1M_2 = M_2M_3 = \&c.$
 $= EN_1 = N_1N_2 = \&c.,$ each of them being $= P$.

Hence, the $\triangle ABC = m$ times the $\triangle ABM_1$; }
 and the $\triangle DEF = n$ times the $\triangle DEN_1$. }

But the $\triangle ABM_1 =$ the $\triangle DEN_1$, because they stand on equal bases and have equal altitudes.

[Bk. II, Th. 2, Note 1.]

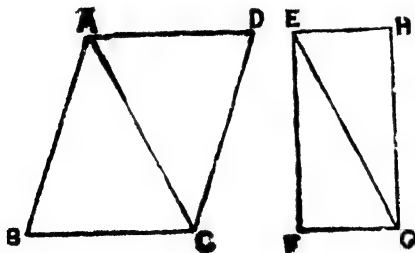
Hence, if a denote the area of either of these two \triangle s, we have

$$\begin{array}{l} \text{the } \triangle ABC = ma \\ \text{and the } \triangle DEF = na \end{array} \quad \left. \vphantom{\begin{array}{l} \text{the } \triangle ABC = ma \\ \text{and the } \triangle DEF = na \end{array}} \right\}$$

$$\text{Hence, } \frac{\text{the } \triangle ABC}{\text{the } \triangle DEF} = \frac{m}{n}.$$

$$\text{and } \therefore \text{ from (1), } \frac{\text{the } \triangle ABC}{\text{the } \triangle DEF} = \frac{BC}{EF};$$

Cor. If two parallelograms have equal altitudes their areas are to one another as their bases.



Let the par^{ms}. AC and EG , standing on the bases BC and FG respectively, have equal altitudes. Join AC, EG . Then the \triangle s ABC, EFG have also equal altitudes.

Now, the par^m. AC : the $\triangle ABC = 2$ }
 and the par^m. EG : the $\triangle EFG = 2$ }

$$\therefore \frac{\text{the par}^m. AC}{\text{the } \triangle ABC} = \frac{\text{the par}^m. EG}{\text{the } \triangle EFG} ;$$

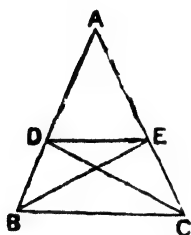
$$\therefore, \text{alternately, } \frac{\text{the par}^m. AC}{\text{the par}^m. EG} = \frac{\text{the } \triangle ABC}{\text{the } \triangle EFG} = \frac{BC}{FG}.$$

Note. In the first diagram, if BC and EF be *incommensurable* either of them may be regarded as a *very large* multiple of a *very small* measure of the other (*Sec. I, Art. 8.*). Hence the str. line P will be indefinitely small, and consequently *m* and *n* will be indefinitely large.

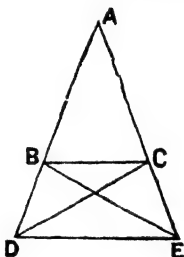
Theorem 2. (EUC. VI. 2.)

If a straight line is drawn parallel to one side of a triangle, it cuts the other two sides, or those sides produced, proportionally.

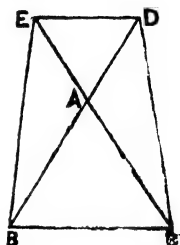
Conversely, if a straight line cuts two sides of a triangle, or the two sides produced, proportionally, then it is parallel to the remaining sides of the triangle.



(1)



(2)



(3)

(i) Let DE be drawn \parallel to the side BC of the $\triangle ABC$, cutting AB , AC , or those sides produced, at D and E .

To prove that $AD : DB = AE : EC$.

Proof. Join BE , CD .

Now, the \triangle s BDE , CED are on the same base DE and between the same \parallel s DE , BC ;

\therefore the $\triangle BDE =$ the $\triangle CED$.

\therefore the $\triangle ADE : \text{the } \triangle BDE = \text{the } \triangle ADE : \text{the } \triangle CED$.

$$\begin{array}{l} \text{But } \frac{\text{the } \triangle ADE}{\text{the } \triangle BDE} = \frac{AD}{DB} \\ \text{and } \frac{\text{the } \triangle ADE}{\text{the } \triangle CED} = \frac{AE}{EC} \end{array} \left. \vphantom{\begin{array}{l} \frac{\text{the } \triangle ADE}{\text{the } \triangle BDE} = \frac{AD}{DB} \\ \frac{\text{the } \triangle ADE}{\text{the } \triangle CED} = \frac{AE}{EC} \end{array}} \right\};$$

(Th. 1.)

$$\therefore \frac{AD}{DB} = \frac{AE}{EC}. \quad \text{Q. E. D.}$$

(ii) Let the str. line DE cut the sides AB, AC, or those sides produced, of the $\triangle ABC$, proportionally; so that $AD : DB = AE : EC$.

To prove that DE is \parallel to BC.

Proof. Join BE, CD.

$$\begin{array}{l} \text{Now,} \\ \text{and} \end{array} \quad \left. \begin{array}{l} \frac{AD}{DB} = \frac{\text{the } \triangle ADE}{\text{the } \triangle BDE} \\ \frac{AE}{EC} = \frac{\text{the } \triangle ADE}{\text{the } \triangle CED} \end{array} \right\} \quad (\text{Th. 1.})$$

$$\text{But} \quad \frac{AD}{DB} = \frac{AE}{EC}; \quad (\text{Hyp.})$$

$$\therefore \frac{\text{the } \triangle ADE}{\text{the } \triangle BDE} = \frac{\text{the } \triangle ADE}{\text{the } \triangle CED};$$

$$\therefore \text{the } \triangle BDE = \triangle CED.$$

But these two \triangle s are on the same base DE and on the same side of it;

$$\therefore DE \text{ is } \parallel \text{ to } BC. \quad (\text{Bk. II, Th. 4, Cor.}) \quad \text{Q. E. D.}$$

C r. 1. If a straight line drawn parallel to the side BC of a triangle ABC cuts the sides AB, AC in D and E, then

$$\frac{AD}{AB} = \frac{AE}{AC}.$$

$$\text{For, in fig. (1),} \quad \frac{AD}{DB} = \frac{AE}{EC},$$

$$\therefore \text{inversely,} \quad \frac{DB}{AD} = \frac{EC}{AE};$$

$$\therefore \frac{DB + AD}{AD} = \frac{EC + AE}{AE} \quad (\text{Componendo}),$$

$$\text{i.e.,} \quad \frac{AB}{AD} = \frac{AC}{AE}.$$

$$\text{Hence, inversely,} \quad \frac{AD}{AB} = \frac{AE}{AC}.$$

Cor. 2. If points D and E be on the sides AB, AC of a triangle ABC, so that $\frac{AD}{AB} = \frac{AE}{AC}$, then DE is parallel to BC.

$$\text{Referring to fig. (1), since } \frac{AD}{AB} = \frac{AE}{AC},$$

$$\therefore \text{ inversely, } \frac{AB}{AD} = \frac{AC}{AE};$$

$$\therefore \quad \frac{AB - AD}{AD} = \frac{AC - AE}{AE} \quad (\text{Dividendo}),$$

$$\text{i.e.,} \quad \frac{DB}{AD} = \frac{EC}{AE}.$$

$$\text{Hence, inversely,} \quad \frac{AD}{DB} = \frac{AE}{EC},$$

and \therefore DE is \parallel to BC.

EXERCISE (29).

1. Prove that the rectangle contained by two lines is a mean proportional between their squares.

2. If two triangles stand on the same base, prove that their areas are to one another as their altitudes.

3. If the medians BE, CF of a given triangle ABC intersect at G, prove that the distances of the points B and C from AG are equal, and hence shew that AG produced bisects BC.

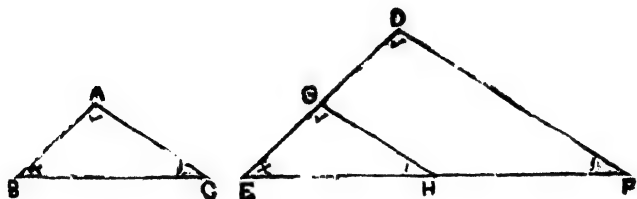
4. ABCD is a trapezium of which AB, DC are the parallel sides. If any straight line EF parallel to AB cuts AD, BC in E and F respectively, prove that $AE : ED = BF : FC$.

5. From a point E in the common base of two triangles ACB, ADB, straight lines are drawn parallel to AC, AD meeting BC, BD in F and G respectively. Prove that FG is parallel to CD.

6. ABC is a triangle ; and DE is drawn parallel to BC, meeting AB, AC in D and E respectively. If BE, CD intersect at F, prove that the triangle ADF is equal to the triangle AEF.

Theorem 3. (Euc. VI. 4.)

If two triangles are equiangular, their corresponding sides are proportional.



Let the $\triangle s$ ABC , DEF be equiangular, having the $\angle s$ A , B , C respectively equal to the $\angle s$ D , E , F .

To prove that $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$.

Proof. Apply the $\triangle ABC$ to the $\triangle DEF$ so that B falls on E and BA on ED , BC falling on the same side of ED as EF .

Then, since the $\angle B =$ the $\angle E$, BC must fall on EF .

Let the pt. G on ED be the new position of A , and the pt. H on EF the new position of C ; then GEH is the new position of the $\triangle ABC$.

Hence, the $\angle EGH =$ the $\angle EDF$, and $\therefore GH$ is \parallel to DF .

Hence, $\frac{EG}{ED} = \frac{EH}{EF}$; (Th. 2, Cor. 1.)

i.e. $\frac{AB}{DE} = \frac{BC}{EF}$.

Similarly, by applying the $\triangle ABC$ to the $\triangle DEF$ so that the $\angle C$ coincides with the $\angle F$, it may be proved that

$$\frac{BC}{EF} = \frac{CA}{FD}.$$

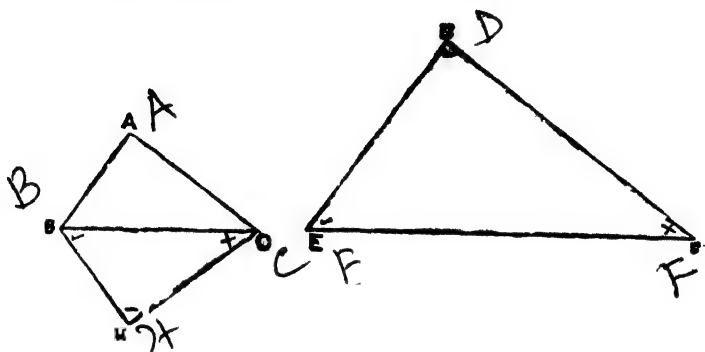
Thus, we have $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$. Q. E. D.

Note. If two triangles are equiangular and also have their *corresponding* sides proportional, then they are said to be **similar**. It is proved that if the first condition be satisfied, the second must *necessarily* follow. Hence, two triangles that are equiangular are necessarily *similar*.

Cor. In a right-angled triangle, if a perpendicular be drawn from the right angle to the hypotenuse, the triangles on each side of it are similar to the whole triangle and to one another.

Theorem 4. (Euc. VI. 5.)

If two triangles have their sides proportional, then they are also equiangular, having those angles equal which are opposite to corresponding sides.



Let the \triangle s ABC, DEF be such that

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}.$$

To prove that the $\angle C =$ the $\angle F$, the $\angle A =$ the $\angle D$, and the $\angle B =$ the $\angle E$.

Proof. Make the $\angle CBH =$ the $\angle E$, and the $\angle BCH =$ the $\angle F$; then the $\angle BHC =$ the $\angle D$.

Now the \triangle s HBC, DEF are equiangular;

$$\therefore \frac{HB}{DE} = \frac{BC}{EF}.$$

But, by hypothesis,

$$\frac{AB}{DE} = \frac{BC}{EF}; \therefore \frac{HB}{DE} = \frac{AB}{DE},$$

and $\therefore HB = AB.$
Similarly, $HC = AC.$

Now in the Δ s ABC , HBC , we have

$$(1) \quad AB = HB.$$

$$(2) \quad AC = HC.$$

and $(3) \quad BC$ common ;

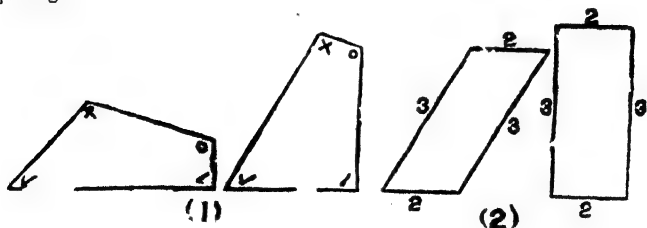
\therefore the two Δ s are congruent.

Hence, the $\angle ACB =$ the $\angle HCB =$ the $\angle F$;
the $\angle BAC =$ the $\angle BHC =$ the $\angle D$;
and the $\angle ABC =$ the $\angle HBC =$ the $\angle E$.

Q. E. D.

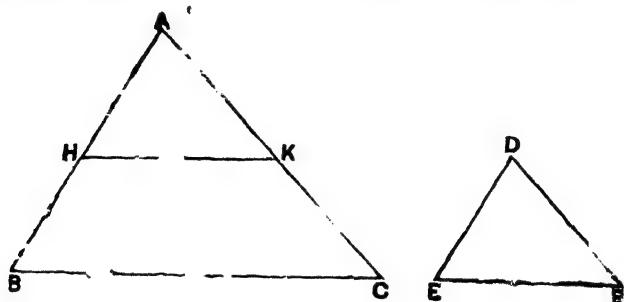
Note 1. From this proposition and the last, it is clear that of the two conditions that must be satisfied in order that two triangles may be *similar*, each necessarily follows from the other. Hence for the similarity of two triangles, it is sufficient that *either* they should be equiangular, *or* they should have their sides proportional.

Note 2. In the case of rectilineal figures other than triangles, however either of the two conditions of similarity does *not* necessarily follow from the other. For instance, in the first of the following diagrams, the two rectilineal figures are equiangular, but their corresponding sides are not proportional; whereas in the second, the two figures have their sides proportional, but they are not equiangular.



Theorem 5. (Euc. VI 6.)

If two triangles have one angle of the one equal to one angle of the other, and the sides about these equal angles proportional, then the triangles are similar.



Let the \triangle s ABC , DEF have the $\angle A = \text{the } \angle D$, and
let $\frac{AB}{DE} = \frac{AC}{DF}$

To prove that the two \triangle s are similar.

Proof. Apply the $\triangle DEF$ to the $\triangle ABC$ so that D falls on A and DE on AB , DF falling on the same side of AB as AC .

Then, since the $\angle A = \text{the } \angle D$, DF must fall on AC .

Let the pt. H on AB and the pt. K on AC be the new positions of E and F respectively; then AHK is the new position of the $\triangle DEF$.

$$\text{Now, } \frac{AB}{DE} = \frac{AC}{DF}; \quad (\text{Hyp.})$$

$$\text{i.e., } \frac{AB}{AH} = \frac{AC}{AK}.$$

$\therefore HK$ is \parallel to BC .

(Th. 2, Cor. 2),

Hence the $\angle AHK$ (which is the $\angle E$) = the $\angle B$; and the $\angle AKH$ (which is the $\angle F$) = the $\angle C$.

Thus, the two \triangle s ABC , DEF are equiangular;
and \therefore their corresponding sides are proportional.

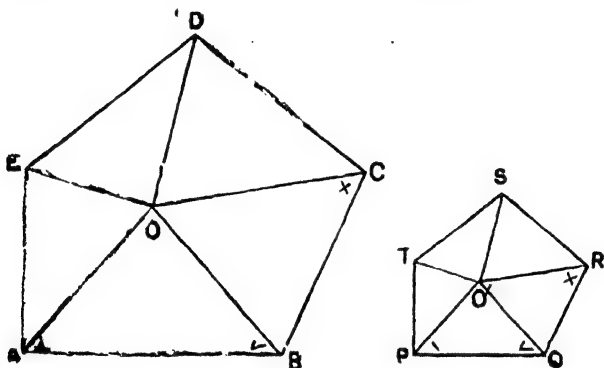
(Th. 3.)

Hence, the \triangle s ABC , DEF are similar.

Q. E. D

Theorem 6.

If a polygon is divided into triangles by lines joining a given point to its vertices, any similar polygon can be divided into corresponding similar triangles.



Let $ABCDE$, $PQRST$ be two similar polygons, having the \angle s A, B, C, D, E respectively equal to the \angle s P, Q, R, S, T ; and having

$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CD}{RS} = \frac{DE}{ST} = \frac{EA}{TP}.$$

Let the polygon $ABCDE$ be divided into the \triangle s AOB, BOC, COD, DOE, EOA by joining any pt. O within the polygon to its vertices.

To prove that there is a point O' within the polygon $PQRST$ such that the $\triangle PO'Q$ is similar to the $\triangle AOB$, the $\triangle QO'R$ is similar to the $\triangle BOC$, and so on.

Proof. Make the $\angle QPO' =$ the $\angle BAO$, and the $\angle PQO' =$ the $\angle ABO$.

Join O'R, O'S, O'T.

(1) The \triangle s AOB, POQ are equiangular;

\therefore they are similar. (Th. 3, Note.)

(2) The $\angle ABC =$ the $\angle PQR$, (Hyp.)

and the $\angle ABO =$ the $\angle PQO$;

\therefore the $\angle OBC =$ the $\angle OQR$ (a)

Also, the \triangle s AOB, PO'Q being similar,

$$\frac{OB}{O'Q} = \frac{AB}{PQ};$$

but $\frac{AB}{PQ} = \frac{BC}{QR}$; (Hyp.)

$\therefore \frac{OB}{O'Q} = \frac{BC}{QR}$ (b)

Hence, from (a) and (b),

the \triangle s BOC, QOR are similar. (Th. 5.)

(3) The $\angle BCD =$ the $\angle QRS$, (Hyp.)

and the $\angle BCO =$ the $\angle QRO'$; (\triangle s BOC, QO'R being equiangular.

\therefore the $\angle OCD =$ the $\angle O'RS$ (c)

Also, the \triangle s BOC, QO'R being similar,

$$\frac{OC}{O'R} = \frac{BC}{QR};$$

but $\frac{BC}{QR} = \frac{CD}{RS}$; (Hyp.)

$\therefore \frac{OC}{O'R} = \frac{CD}{RS}$ (d)

Hence, from (c) and (d),

the \triangle s COD, RO'S are similar. (Th. 5.)

In the same way,

the \triangle s DOE, SO'T are similar ;

and so are the \triangle s EOA, TO'P.

Q. E. D.

Note. If the pt. O coincides with one of the angular points of the figure **ABCDE**, then O' will coincide with the corresponding angular point of the figure **PQRST**. For instance, if O coincides with A, O' will coincide with P.

EXERCISE (30).

1. ABC is a triangle, and DE is drawn parallel to BC, meeting AB, AC in D and E respectively. If BE, CD intersect in O, prove that the area of the triangle COE is a mean proportional between the areas of the triangles DOE and BOC.

2. B is the mid-point of a straight line AC, and the triangles ADB, BEC are between the same parallels. If a straight line parallel to BC cuts DA, DB, EB, EC in F, G, H, K respectively, prove that $FG = HK$.

3. If the numerical measures of the sides of one triangle be 2, 3, 4, and those of another, 6, 9, 12, prove that the two triangles are equiangular.

4. ABC is a triangle ; BE, CF are drawn to the opposite sides so as to intersect at G on the median from A. Prove that the $\triangle AGC : \text{the } \triangle BGC = AF : FB$, and hence prove that FE is parallel to BC.

5. AB and CD are two parallel straight lines, and E is the mid-point of CD ; AC and BE meet at F, and AE and BD meet at G. Prove that FG is parallel to AB.

6. Two straight lines AB, CD intersect at B , so that $OA : OC = OD : OB$. Prove that the $\angle ADO =$ the $\angle CBO$, and hence shew that the four points A, B, C, D are concyclic.

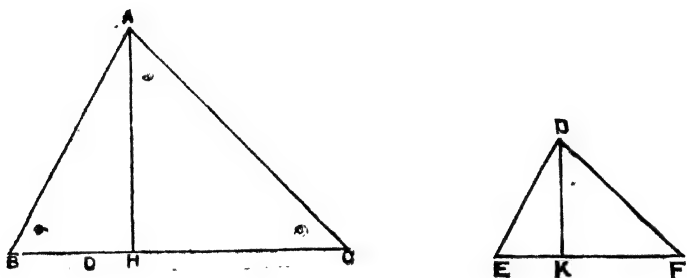
7. In a $\triangle ABC$, AD is drawn perpendicular to BC . If AD is the mean proportional to BD and DC , prove that the $\angle BAC$ is a rt. \angle .

8. AC and BD are drawn perpendiculars to a given straight line CD from two given points A and B ; AD and BC intersect in E , and EF is drawn perpendicular to CD . Prove that AF and BF are equally inclined to CD .

9. If a polygon is divided into triangles having one of the vertices of the polygon for a common vertex, shew that any similar polygon can be divided into corresponding similar triangles.

Theorem 7. (Euc. VI. 19.)

The ratio of the areas of two similar triangles is equal to the ratio of the squares on the corresponding sides.



Let ABC , DEF be two similar Δ s having the \angle s A, B, C , respectively equal to the \angle s D, E, F ; so that BC and EF are a pair of corresponding sides.

To prove that

$$\frac{\Delta ABC}{\Delta DEF} = \frac{BC^2}{EF^2}.$$

Proof. Let AH , DK be \perp s to BC , EF ; and let a, d, h, k be the numerical measures of BC, EF, AH, DK respectively, referred to one and the same unit.

Then the area of the $\Delta ABC = \frac{1}{2} ah$ units of area ;
and the area of the $\Delta DEF = \frac{1}{2} dk$ units of area. }

$$\therefore \frac{\Delta ABC}{\Delta DEF} = \frac{ah}{dk} = \frac{a}{d} \cdot \frac{h}{k}.$$

Now, the rt. $\angle B =$ the $\angle E$,
and the rt. $\angle AHB =$ the rt. $\angle DKE$; }

\therefore the Δ s AHB, DKE are equiangular.

Hence $\frac{AH}{DK} = \frac{AB}{DE};$

but $\frac{AB}{DE} = \frac{BC}{EF};$ (Hyp).

$\therefore \frac{AH}{DK} = \frac{BC}{EF},$ and $\therefore \frac{h}{k} = \frac{a}{d}.$

Hence, $\frac{\triangle ABC}{\triangle DEF} = \frac{a}{d} \cdot \frac{a}{d} = \frac{a^2}{d^2}.$

But a^2 = number of units of area in the square on BC,)
and d^2 = number of units of area in the square on EF,)

$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{BC^2}{EF^2}.$ Q. E. D.

Note 1. Let $a, b, c, d, e, f,$ be the numerical measures of $BC, CA, AB, EF, FD, DE,$ respectively referred to one and the same unit.

Then, since $\frac{BC}{EF} = \frac{CA}{FD} = \frac{AB}{DE}$ we must have $\frac{a}{d} = \frac{b}{e} = \frac{c}{f};$

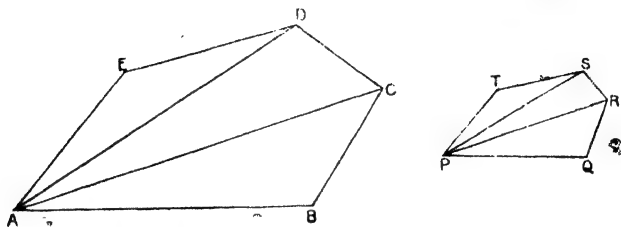
and $\therefore \frac{a^2}{d^2} = \frac{b^2}{e^2} = \frac{c^2}{f^2}.$ Hence, $\frac{BC^2}{EF^2} = \frac{CA^2}{FD^2} = \frac{AB^2}{DE^2}.$

Hence $\frac{\triangle ABC}{\triangle DEF} = \frac{CA^2}{FD^2}$ and also $= \frac{AB^2}{DE^2}.$

Note 2. That $\frac{\triangle ABC}{\triangle DEF} = \frac{CA^2}{FD^2}$ may also be proved independently by drawing perpendiculars from B and E on AC and DF respectively.

Theorem 8. (EUC. VI. 20.)

The ratio of the areas of two similar polygons is equal to the ratio of the squares on the corresponding sides.



Let $ABCDE$, $PQRST$, be two similar polygons, having the \angle s A, B, C, D, E respectively equal to the \angle s P, Q, R, S, T ; so that AB and PQ are a pair of corresponding sides.

To prove that

$$\frac{\text{fig. } ABCDE}{\text{fig. } PQRST} = \frac{AB^2}{PQ^2}.$$

Proof. Join AC, AD, PR, PS .

Then the \triangle s ABC, PQR are similar.
 the \triangle s ACD, PRS are similar;
 and the \triangle s ADE, PST are similar. } (Th. 6.)

Hence
$$\left. \begin{array}{l} \frac{\triangle ABC}{\triangle PQR} = \frac{AC^2}{PR^2}; \\ \frac{\triangle ACD}{\triangle PRS} = \frac{AC^2}{PR^2}; \end{array} \right\} \quad (\text{Th. 7.})$$

$$\therefore \frac{\triangle ABC}{\triangle PQR} = \frac{\triangle ACD}{\triangle PRS} \quad \dots \dots \dots (1)$$

$$\text{Similarly, } \frac{\triangle ACD}{\triangle PRS} = \frac{\triangle ADE}{\triangle PST} \quad \dots \dots \dots (2)$$

Hence, from (1) and (2),

$$\frac{\triangle ABC}{\triangle PQR} = \frac{\triangle ACD}{\triangle PRS} = \frac{\triangle ADE}{\triangle PST}.$$

Therefore each of these ratios

$$\begin{aligned}
 &= \frac{\triangle ABC + \triangle ACD + \triangle ADE}{\triangle PQR + \triangle PRS + \triangle PST} \quad (\text{Addendo}) \\
 &= \frac{\text{fig. } ABCDE}{\text{fig. } PQRST}
 \end{aligned}$$

$$\text{Hence, } \frac{\text{fig. } ABCDE}{\text{fig. } PQRST} = \frac{\triangle ABC}{\triangle PQR} = \frac{AB^2}{PQ^2} \quad \text{Q. E. D.}$$

EXERCISE (31).

1. ABC is a triangle, and D is the mid-pt. of the semi-circle described on AB as diameter; E is a point on AB so that $AE = AD$. Prove that the straight line drawn through E parallel to BC bisects the triangle.

2. Shew how to draw a line parallel to the base of a triangle so as to form with the other two sides produced a triangle double of the given triangle.

3. ABC is a right-angled triangle of which BC is the hypotenuse; AD is drawn perpendicular to BC . Prove that $BD : DC = AB^2 : AC^2$.

4. ABC is a triangle inscribed in a circle, and the tangent at A meets BC produced in D . Prove that

$$BD : CD = AB^2 : AC^2.$$

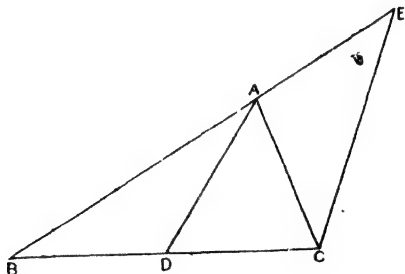
5. BE , CF are the perpendiculars on the sides CA , AB of a triangle ABC . Prove that the $\triangle ABC$: the $\triangle AFE = AB^2 : AE^2$.

6. If two regular pentagons be inscribed in two given circles, prove that their areas are to one another as the squares on the radii of the circles.

Theorem 9. (Euc. VI. 3.)

The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.

Conversely, if one side of a triangle is divided internally in the ratio of the other two sides, then the line drawn from the point of division to the opposite vertex is the internal bisector of the angle at that vertex.



(i) In the $\triangle ABC$, let AD be the internal bisector of the $\angle BAC$, meeting BC in D .

To prove that $\frac{BD}{DC} = \frac{AB}{AC}$.

Proof. Let CE be \parallel to DA , meeting BA produced at E .

Then the $\angle AEC =$ the ext. $\angle BAD$; }
and the $\angle ACE =$ the alt. $\angle CAD$. }

But the $\angle BAD =$ the $\angle CAD$; (Hyp.)

\therefore the $\angle AEC =$ the $\angle ACE$.

\therefore $AC = AE$ (a)

Now, in the $\triangle BCE$, DA is \parallel to CE ;

$$\therefore \frac{BD}{DC} = \frac{BA}{AE} . \quad (Th. 2.)$$

$$\text{Hence, from } (\alpha), \frac{BD}{DC} = \frac{BA}{AC} . \quad \text{Q. E. D.}$$

(ii) In the $\triangle ABC$, let the side BC be divided internally at D so that $\frac{BD}{DC} = \frac{BA}{AC}$; and join AD .

To prove that the $\angle BAD =$ the $\angle CAD$.

Proof. Let CE be \parallel to DA , meeting BA produced in E .

Then in the $\triangle BCE$, DA is \parallel to CE ;

$$\therefore \frac{BD}{DC} = \frac{BA}{AE} \quad (Th. 2.)$$

$$\text{But } \frac{BD}{DC} = \frac{BA}{AC} ; \quad (Hyp.)$$

$$\therefore \frac{BA}{AC} = \frac{BA}{AE} , \text{ and } \therefore AC = AE.$$

$$\text{Hence the } \angle AEC = \text{the } \angle ACE. \quad \dots \dots (\beta)$$

Now, since DA is \parallel to CE ,

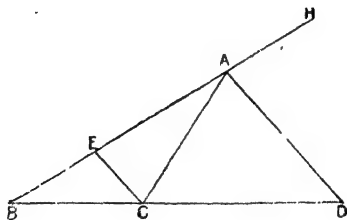
$$\therefore \left. \begin{array}{l} \text{the } \angle AEC = \text{the ext. } \angle BAD ; \\ \text{and the } \angle ACE = \text{the alt } \angle CAD. \end{array} \right\}$$

Hence, from (β) ,

$$\text{the } \angle BAD = \text{the } \angle CAD. \quad \text{Q. E. D.}$$

Theorem 10. (Euc. VI. A.)

The external bisector of an angle of a triangle divides the opposite side externally in the ratio of the sides containing the angle. Conversely, if one side of a triangle is divided externally in the ratio of the other two sides, then the line drawn from the point of division to the opposite vertex is the external bisector of the angle at that vertex.



(i) In the $\triangle ABC$, let H be any point on BA produced and let AD be the bisector of the $\angle HAC$, meeting BC produced in D ; then AD is the *external* bisector of the $\angle BAC$.

To prove that $\frac{BD}{DC} = \frac{BA}{AC}$.

Proof. Let $CE \parallel$ to DA , meeting BA in E .

Then the $\angle AEC =$ the ext. $\angle HAD$; }
and the $\angle ACE =$ the alt. $\angle CAD$. }

But the $\angle HAD =$ the $\angle CAD$, (Hyp.)

\therefore the $\angle AEC =$ the $\angle ACE$;

$\therefore AC = AE$ (a)

Now, in the $\triangle BCE$, $DA \parallel$ to CE ;

$\therefore \frac{BD}{DC} = \frac{BA}{AE}$. (Th. 2.)

Hence, from (α), $\frac{BD}{DC} = \frac{BA}{AC}$. Q. E. D.

(ii) In the $\triangle ABC$, let the side BC be divided externally at D so that $\frac{BD}{DC} = \frac{BA}{AC}$.

Join AD and produce BA to any point H .

To prove that AD is the bisector of the $\angle HAC$.

Proof. Let CE be \parallel to DA , meeting BA in E .

Then in the $\triangle BCE$, DA is \parallel to CE ;

$$\therefore \frac{BD}{DC} = \frac{BA}{AE}. \quad (Th. 2.)$$

$$\text{But} \quad \frac{BD}{DC} = \frac{BA}{AC}; \quad (Hyp.)$$

$$\therefore \frac{BA}{AC} = \frac{BA}{AE}, \text{ and } \therefore AC = AE.$$

Hence the $\angle AEC =$ the $\angle ACE$ (β)

Now, since DA is \parallel to CE ,

$$\left. \begin{array}{l} \therefore \text{ the } \angle AEC = \text{ the ext. } \angle HAD; \\ \text{ and the } \angle ACE = \text{ the alt. } \angle CAD. \end{array} \right\}$$

Hence, from (β),

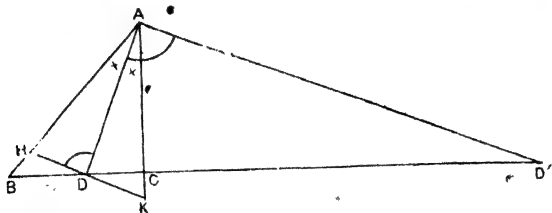
$$\text{the } \angle HAD = \text{ the } \angle CAD;$$

i.e., AD is the bisector of the $\angle HAC$. Q. E. D.

Cor. If the side BC of a triangle ABC be divided internally at D and externally at D' in the ratio of BA to AC then DD' subtends a right angle at A .

Note 1. If a straight line BC is divided internally at D and externally at D' in the same ratio (so that $BD : DC = BD' : D'C$),

and if DD' subtends a right angle at any point A , then it may be proved that AD is the internal bisector of the angle BAC .



Let HDK be drawn \parallel to AD' , meeting AB in H and AC produced in K . Since HK is \parallel to AD' , \therefore the $\angle ADH =$ the $\angle DAD' =$ a rt. \angle .

Since $\frac{BD}{DC} = \frac{BD}{D'C}$, \therefore , alternately $\frac{BD}{BD'} = \frac{DC}{D'C}$.

Now the Δ s BDH , $BD'A$ being equiangular, we have

$\frac{BD}{BD'} = \frac{DH}{D'A}$; and the Δ s DCk , $D'CA$ being equiangular, we have

$\frac{DC}{D'C} = \frac{DK}{D'A}$. Hence $\frac{DH}{D'A} = \frac{DK}{D'A}$, and $\therefore DH = DK$. Now it is easy to see that the Δ s ADH , ADK are congruent; and \therefore the $\angle BAD =$ the $\angle CAD$.

Note 2. When a given straight line is divided internally and externally in the same ratio, it is said to be cut **harmonically**. Hence any side of a triangle may be said to be *cut harmonically* by the internal and external bisectors of the opposite vertical angle.

EXERCISE (32).

1. $ABCD$ is a quadrilateral, and the bisectors of the angles A and C meet on the diagonal BD . Prove that the bisectors of the angles B and D meet on AC .

2. In a given circle, AB is a diameter and CD is a chord perpendicular to it; through any point E in CD , AE and BE are drawn to meet the circle in F and G . Prove that the quadrilateral $CFDG$ has any two of its adjacent sides in the same ratio as the other two.

3. ABC is a triangle, and BO , CO are the bisectors of the angles B and C . If AO produced meets BC in D , prove that $AB : AC = BD : CD$.

4. ABC is a triangle, and BP , CP are the bisectors of the exterior angles at B and C . If AP cuts BC at D , prove that $AB : AC = BD : DC$.

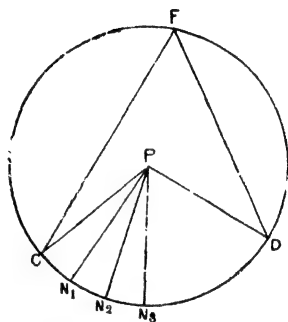
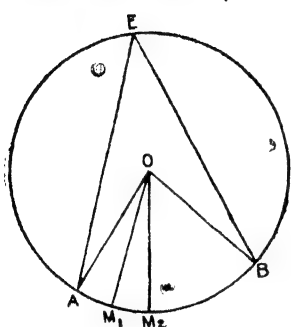
5. In a given circle, AB is a diameter, and P is any point on the circumference; PC , PD are two straight lines on opposite sides of PA and equally inclined to it, meeting AB in C and D . Prove that $AC : CB = AD : DB$.

6. ABC is a triangle, and AD is the bisector of the angle A , meeting BC in D . Prove that $AB : BD = AB + AC : BC$.

Hence, if O be the in-centre of the triangle ABC , prove that $AO : OD = AB + AC : BC$.

Theorem 11. (Euc. VI. 33.)

In equal circles, angles, whether at the centres or at the circumferences, have the same ratio as the arcs on which they stand,



Let ABE, CDF be two equal \odot s. of which the centres are O and P.

(i) the \angle s AOB, CPD at the centres stand on the arcs AB, CD.

To prove that $\frac{\text{the } \angle \text{AOB}}{\text{the } \angle \text{CPD}} = \frac{\text{the arc AB}}{\text{the arc CD}}$.

Proof. Suppose that the m th part of the arc AB is contained n times in the arc CD; i.e., suppose that a common measure of the two arcs is contained m times in the arc AB, and n times in the arc CD.

$$\text{Hence } \frac{\text{the arc AB}}{\text{the arc CD}} = \frac{m}{n}.$$

Let the arc AB be divided into m equal parts at the pts. M_1, M_2 , &c.; and let the arc CD be divided into n equal parts at N_1, N_2, N_3 , &c.

Join OM_1, OM_2 , &c., PN_1, PN_2, PN_3 , &c.,

Then the arcs AM_1 , M_1M_2 , &c., CN_1 , N_1N_2 , N_2N_3 , &c. are all equal; and \therefore the \angle s subtended by them at the centres are all equal to one another. *

Hence, if each of the \angle s AOM_1 , M_1OM_2 , &c., CPN_1 , N_1PN_2 , &c. be denoted by α , we have

$$\left. \begin{array}{l} \text{the } \angle AOB = m\alpha, \\ \text{and the } \angle CPD = n\alpha; \end{array} \right\}$$

$$\therefore \frac{\text{the } \angle AOB}{\text{the } \angle CPD} = \frac{m}{n},$$

$$\text{and is } \therefore = \frac{\text{the arc AB}}{\text{the arc CD}}. \quad \text{Q. E. D.}$$

(ii) Let the \angle s AEB , CFD at the circumferences stand on the arcs AB , CD .

$$\text{To prove that } \frac{\text{the } \angle AEB}{\text{the } \angle CFD} = \frac{\text{the arc AB}}{\text{the arc CD}}.$$

Proof. Join AO , OB , CP , PD .

$$\left. \begin{array}{l} \text{Then the } \angle AOB = \text{twice the } \angle AEB \\ \text{and the } \angle CPD = \text{twice the } \angle CFD. \end{array} \right\}$$

$$\therefore \frac{\text{the } \angle AOB}{\text{the } \angle AEB} = \frac{\text{the } \angle CPD}{\text{the } \angle CFD} \text{ (each being } = 2)$$

and \therefore alternately,

$$\frac{\text{the } \angle AOB}{\text{the } \angle CPD} = \frac{\text{the } \angle AEB}{\text{the } \angle CFD}.$$

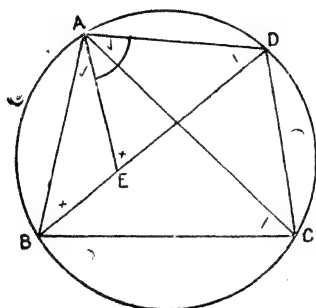
$$\text{But } \frac{\text{the } \angle AOB}{\text{the } \angle CPD} = \frac{\text{the arc AB}}{\text{the arc CD}} \text{ (Proved above);}$$

$$\therefore \frac{\text{the } \angle AEB}{\text{the } \angle CFD} = \frac{\text{the arc AB}}{\text{the arc CD}}. \quad \text{Q. E. D.}$$

Note. From Book III, Th. 4, Cor. 3, it is known that in equal circles sectors standing on equal arcs are equal. Hence it is easy to see that the sector AOB : the sector CPD = the arc AB : the arc CD .

Theorem 12. (Euc. VI. D.)

The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the two rectangles contained by its opposite sides.



Let ABCD be a quadr. inscribed in a \odot , and let AC, BD be its diagonals.

To prove that $AC \cdot BD = AB \cdot DC + AD \cdot BC$.

Proof. Make the $\angle BAE =$ the $\angle CAD$, and let AE meet BD in E.

Then in the \triangle s BAE, CAD,

the $\angle BAE =$ the $\angle CAD$;

and the $\angle ABE =$ the $\angle ACD$. (\angle s in the same segment.)

Hence the two \triangle s are equiangular;

$$\therefore \frac{AB}{AC} = \frac{BE}{DC}; \quad (\text{Th. 3.})$$

and $\therefore AB \cdot DC = AC \cdot BE$. (Sec. I, Art, 18.) ... (a)

Again, adding the $\angle EAC$ to each of the equal \angle s BAE, CAD, we have

the $\angle BAC =$ the $\angle EAD$;

also the $\angle ACB = \text{the } \angle ADE$.

($\angle s$ in the same segment.)

Hence the $\triangle s$ ACB, ADE are equiangular;

$$\therefore \frac{AD}{AC} = \frac{DE}{BC},$$

and $\therefore AD \cdot BC = AC \cdot DE$ (β)

Hence, from (α) and (β),

$$\begin{aligned} AB \cdot DC + AD \cdot BC &= AC \cdot BE + AC \cdot DE \\ &= AC(BE + DE) \quad (Bk. II, Th. 7). \end{aligned}$$

$$= AC \cdot BD. \quad \text{Q. E. D.}$$

Note. This proposition is known as **Ptolemy's Theorem**.

EXERCISE (33).

1. ABC is an equilateral triangle inscribed in a given circle, and P is any point on the arc BC . Prove that $PA = PB + PC$.

2. If the diagonals of a cyclic *quadrilateral be perpendicular to each other, prove that the sum of the two rectangles contained by its opposite sides is double the area of the quadrilateral.

3. ABC is a triangle; AD is the perpendicular from A on BC , and AP is a diameter of the circum- \odot of the given triangle. Prove that $AP \cdot AD = AB \cdot AC$.

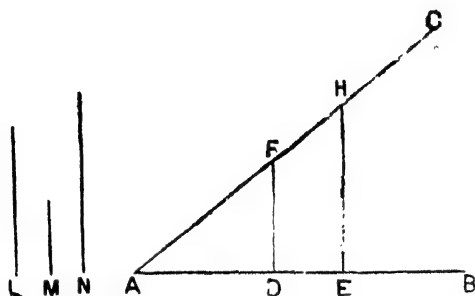
Hence, if a, b, c, γ be the numerical measures of the sides and the circum-radius respectively of any given triangle, and if S be the numerical measure of its area, prove that $\gamma = abc \div 4S$.

SECTION III.

PROBLEMS.

Problem 1.

To find the fourth proportional to three given straight lines.



Let L, M, N be the three given str. lines.

It is required to find a str. line X such that $L : M = N : X$.

Cons. Draw two str. lines AB, AC, forming an angle and unlimited towards B and C.

From AB cut off $AD = L$, and $DE = M$; and from AC cut off $AF = N$.

Join DF; and draw $EH \parallel$ to DF, meeting AC at H.

Then FH is the reqd. str. line.

Proof. In the $\triangle AEH$, DF is \parallel to EH ;

$$\therefore \quad , \quad \frac{AD}{DE} = \frac{AF}{FH} , \quad (Th \ 2.)$$

But $AD = L$, $DE = M$, and $AF = N$.

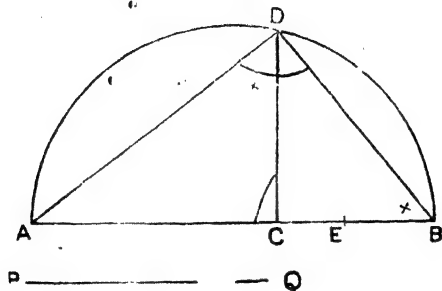
$$\therefore \quad \frac{L}{M} = \frac{N}{FH} ;$$

i.e., FH is the fourth proportional to L , M , N . Q. E. F.

Note. By a similar construction we can find the *third* proportional to two given straight lines. For if L , M be the two given straight lines, the only change in the construction will be to make $AF = M$; and then FH will be such that $L : M = M : FH$.

Problem 2.

To find the mean proportional to two given straight lines.



Let AB, PQ be the two given straight lines.

It is required to find a str. line X such that $AB : X = X : PQ$.

Cons If AB is the greater of the two given str. lines, from AB cut off $AC = PQ$; and on AB as diameter describe a semi- \odot .

Draw $CD \perp$ to AB meeting the semi- \odot at D .

Join AD , and from AB cut off $AE = AD$.

Then AE is the reqd. str. line.

Proof. Join DB .

The $\angle ADB$, being in a semi- \odot , is a rt. \angle ;

\therefore the $\angle ADB =$ the $\angle ACD$;

also the $\angle ABD =$ the $\angle ADC$, each of them being complementary to the $\angle BAD$.

Hence the \triangle s ADB, ACD are equiangular;

$$\therefore \frac{AB}{AD} = \frac{AB}{AC} \quad (Th. 3.)$$

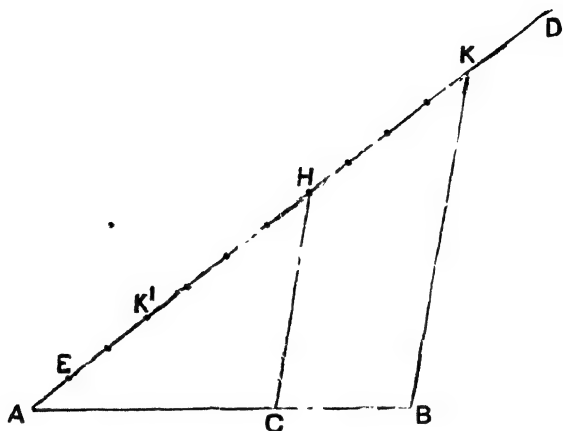
But $AD = AE$, and $AC = PQ$,

$$\therefore \frac{AB}{AE} = \frac{AE}{PQ};$$

i.e., AE is the mean proportional to AB and PQ . Q. E. F.

Problem 3.

To divide a given straight line internally in a given ratio (say in the ratio of 7 to 4).



Let AB be the given str. line.

It is required to find a pt. C in AB such that $\frac{AC}{CB} = \frac{7}{4}$.

Cons. Draw a str. line AD , making an angle with AB , and unlimited towards D .

From AD cut off any convenient length AE , and measure off $AH = 7$ times AE .

Along HD measure off $HK = 4$ times AE .

Join KB ; and draw $HC \parallel$ to KB , meeting AB in C .

Then C is the reqd. pt.

Proof. Evidently $\frac{AH}{HK} = \frac{7}{4}$.

Now in the $\triangle ABK$, CH is \parallel to BK ;

$$\therefore \frac{AC}{CB} = \frac{AH}{HK} \quad (\text{Th. 2.})$$

$$= \frac{7}{4} \quad \text{Q. E. F.}$$

Note 1. If HK' equal to 4 times AE , be measured off along HA ; if $K'B$ be joined; and if HC' be drawn parallel to $K'B$, meeting AB produced in C ; then AB is divided *externally* at C in the ratio of 7 to 4. This can be easily proved as above.

Note 2. $AH = 7$ times AE , and $AK = 11$ times AE ;

$\therefore \frac{AH}{AK} = \frac{7}{11}$. But the $\triangle s$ ACH , ABK being equiangular, we have

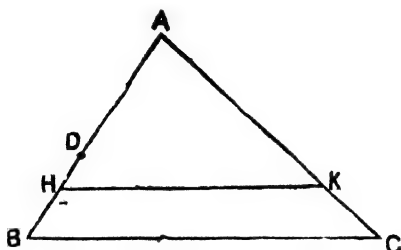
$$\frac{AC}{AB} = \frac{AH}{AK}. \quad \text{Hence } \frac{AC}{AB} = \frac{7}{11}, \text{ i.e., } AC \text{ is } \frac{7}{11} \text{ of } AB.$$

In the same way, it is easy to see that $AC' = \frac{7}{3} AB$, where C' is found in the manner indicated in Note 1. Thus, by a construction similar to that of this Proposition, we can find *any* given fraction of a given straight line.

Note 3. There is only one point in which a given straight line may be divided internally (or externally) in a given ratio. For, if C be a point on AB such that $\frac{AC}{CB} = \frac{7}{4}$, then AC is $\frac{7}{11}$ of AB . Similarly, if P be a point on AB such that $\frac{AP}{PB} = \frac{7}{4}$, then AP is also $= \frac{7}{11}$ of AB . Hence $AP = AC$, and $\therefore P$ coincides with C .

Problem 4.

From a given triangle to cut off any given fraction (say $\frac{3}{5}$) of it by a straight line drawn parallel to one of its sides.



Let ABC be a given \triangle , and let BC be the side considered.

It is required to draw a straight line HK parallel to BC so that the triangle AHK may be $=\frac{3}{5}$ of the triangle ABC .

Cons. Find a pt. D in AB such that

$$\frac{AD}{AB} = \frac{3}{5}. \quad (\text{Prob. 3.})$$

To AD and AB find the mean proportional AH , so that $AH^2 = AD \cdot AB$. (Prob. 2.)

Draw $HK \parallel$ to BC , meeting AC in K .

Then HK is the reqd. str. line.

Proof. The \triangle s AHK , ABC are equiangular, and \therefore similar.

$$\begin{aligned} \text{Hence, } \frac{\triangle AHK}{\triangle ABC} &= \frac{AH^2}{AB^2} & (\text{Th. 7.}) \\ &= \frac{AD \cdot AB}{AB^2}. \end{aligned}$$

But, since 3 and 5 are the numerical measures of AC and AB, referred to one and the same unit, we have

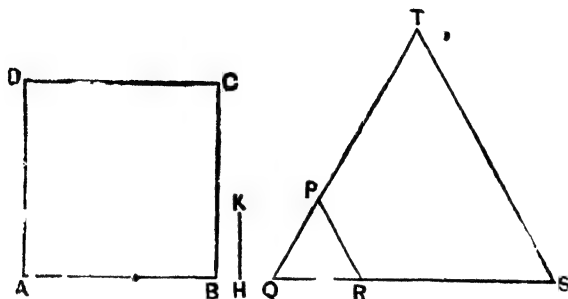
the rect. $AD \cdot AB = 3 \times 5$ units of area, }
and the sq. on $AB = 5 \times 5$ units of area ; }

$$\therefore \frac{AD \cdot AB}{AB^2} = \frac{3 \times 5}{5 \times 5} = \frac{3}{5}.$$

$$\text{Hence, } \frac{\triangle AHK}{\triangle ABC} = \frac{3}{5}. \quad \text{Q. E. F.}$$

Problem 5.

To construct an equilateral triangle equal to a given square.



Let ABCD be the given square.

It is required to construct an equilateral triangle equal to the sq. ABCD.

Cons. Describe an equilat. $\triangle PQR$. Find the str. line HK which is a side of the sq. equal to the $\triangle PQR$.

(Bk. II. Prob. 3, Note.)

From QR, produced, if necessary, cut off QS = the fourth proportional to HK, AB, QR. (Prob. 1.)

Produce QP to T making QT = QS; join TS.

Then TQS is the equilat. \triangle reqd.

Proof. We have $\frac{HK}{AB} = \frac{QR}{QS}$; (Cons.)

$$\therefore \frac{HK^2}{AB^2} = \frac{QR^2}{QS^2}. \quad (\text{Sec. 1, Art. 24.})$$

Now, the \triangle s PQR and TQS are similar;

$$\therefore \frac{\triangle PQR}{\triangle TQS} = \frac{QR^2}{QS^2} = \frac{HK^2}{AB^2}.$$

But

$$HK^2 = \text{the } \triangle PQR; \quad (\text{Cons.})$$

\therefore

$$\frac{\triangle PQR}{\triangle TQS} = \frac{\triangle PQR}{\text{sq. } ABCD}.$$

Hence the equilat. $\triangle TQS = \text{the sq. } ABCD.$ Q. E. F.

EXERCISE (34).

1. On a given straight line construct a rectangle equal to a given rectangle.

2. Construct a square equal to a given rectangle.

3. Divide a given straight line internally in the ratio of 8 to 5.

4. Divide a given straight line externally in the ratio of 5 to 8.

5. AB is a given straight line. Find a straight line PQ such that

$$\frac{PQ^2}{AB^2} = \frac{2}{3}.$$

6. AB is a given straight line. Find a straight line MN such that

$$\frac{MN^2}{AB^2} = \frac{9}{5}.$$

7. Trisect a given triangle by lines drawn parallel to its base.

8. Construct a triangle similar to a given triangle and equal in area to twice the given triangle.

9. ABCD is a given quadrilateral, and PQ, a given straight line. On PQ construct a similar quadrilateral so that AB and PQ may be a pair of corresponding sides.

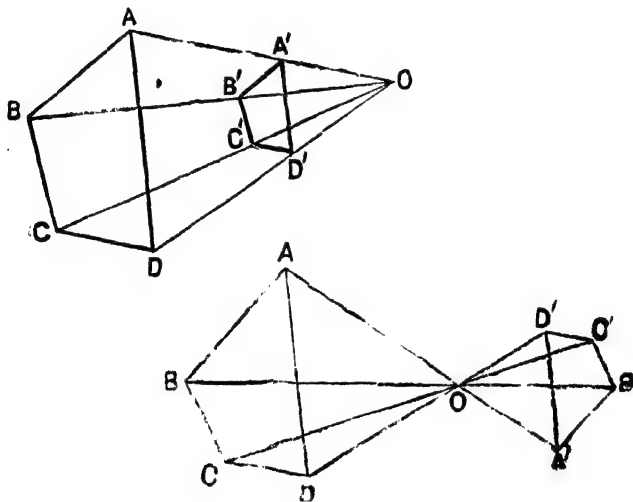
Hence shew how to construct a quadrilateral similar to the quadrilateral ABCD and equal in area to one-third of it.

10. Construct a triangle, having given the base, the vertical angle and the ratio of the sides.

SECTION IV.

MISCELLANEOUS PROPOSITIONS.

1. *If two unequal similar figures are so placed that their corresponding sides are parallel, then the lines joining their corresponding angular points are concurrent.*



Let $ABCD, A'B'C'D'$ be two unequal similar figures, of which the angular points A', B', C', D' respectively correspond to the pts. A, B, C, D .

Let the figures be so placed that their corresponding sides are parallel.

To prove that AA', BB', CC', DD' are concurrent.

Proof. Let AA', BB' meet in O .

Then $A'B'$ being \parallel to AB , the \triangle s AOB , $A'OB'$ are equiangular;

$$\therefore \frac{BO}{B'O} = \frac{AB}{A'B'}$$

Similarly, if CC' meets BB' in P , we have

$$\frac{BP}{B'P} = \frac{BC}{B'C'}$$

$$\text{But} \quad \frac{AB}{A'B'} = \frac{BC}{B'C'} \quad (\text{Hyp.})$$

$$\therefore \frac{BO}{B'O} = \frac{BP}{B'P}$$

and $\therefore P$ coincides with O . (*Prob. 3, Note 3.*)

Thus, CC' passes through the pt. where AA' and BB' intersect, *i.e.*, through O .

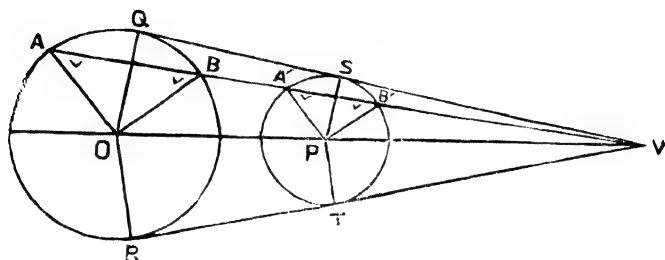
In the same way it can be proved that DD' passes through the pt. where BB' and CC' intersect, and \therefore through O .

Hence, AA' , BB' , CC' , DD' all pass through the same point O . Q. E. D.

Note. The point O is called a centre of similarity of the two figures $ABCD$, $A'B'C'D'$.

2, *If a direct common tangent to two given unequal circles meets the line of centres in V , then the other direct common tangent, as also the line joining the*

extremities of any two parallel radii of the circles drawn in the same sense, will pass through V.



Let O, P be the centres of the given \odot s and QS a direct common tangent which meets OP in V .

Let RT be the other direct common tangent ; and let OA, PA' be *any* two parallel radii drawn in the same sense. Join AA' .

To prove that RT and AA' both pass through V .

Proof. The \angle s OQV, PSV being rt. \angle s, OQ is \parallel to PS ;

\therefore the \triangle s OQV, PSV are equiangular ;

$$\therefore \frac{OV}{PV} = \frac{OQ}{PS}.$$

Thus, V divides OP *externally* in the ratio of the radii of the \odot s.

Similarly, if RT meets OP in H , it can be proved that H divides OP externally in the ratio of the radii.

Hence H is no other pt. than V ; (*Prob. 3, Note 3.*)
i.e., AT passes through V .

Again if AA' meets OP in K , the Δ s $OA'K$, PAK are evidently equiangular ;

$$\therefore \frac{OK}{PK} = \frac{OA}{PA}.$$

Thus, K divides OP externally in the ratio of the radii ; and $\therefore K$ coincides with V .

Hence AA' passes through V .

Q. E. D.

Note 1. If AA' cuts the circles again in B and B' , it is easy to see that OB and PB' are parallel. Hence the Δ s OBV , $PB'V$ are equiangular ; the Δ s OAV , $PA'V$ are equiangular too.

$$\text{Hence, } \frac{VA}{VA'} = \frac{VO}{VP} = \frac{VB}{VB'} ; \therefore VA.VB' = VA'.VB.$$

Note 2. $VA'.VB' = VS^2$; (Bk. III, Th. 17, Cor. 1.)

$\therefore \frac{VA'}{VS} = \frac{VS}{VB'}$. Again, the Δ s VSP , VQO are equiangular and so are the Δ s $VB'P$, VBO ;

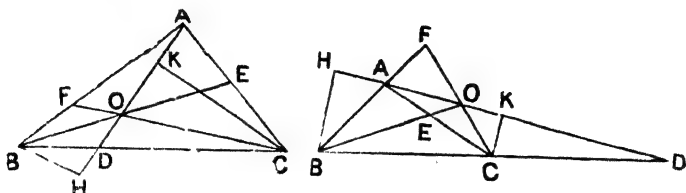
$$\therefore \frac{VS}{VQ} = \frac{VP}{VO} = \frac{VB'}{VB} ; \therefore \text{alternately, } \frac{VS}{VB'} = \frac{VQ}{VB}.$$

$$\text{Hence, } \frac{VA'}{VS} = \frac{VQ}{VB} \therefore VS.VQ = VA'.VB.$$

Note 3. It can be easily proved, as in this Proposition, that if a transverse common tangent to two given circles meets the line of centres in W , then the other transverse common tangent, as also the line joining the extremities of any two parallel radii of the circles drawn in opposite senses, will pass through W .

Note 4. The point where the transverse common tangents to two given circles meet the line of centres is called the **internal centre of similitude** and the point where the direct common tangents meet the line of centres is called the **external centre of similitude** of the two circles. These two points respectively divide the line joining the centres *internally* and *externally* in the ratio of the radii of the circles.

3. *If three concurrent straight lines are drawn from the vertices of a triangle to meet the opposite sides, then the product of three alternate segments, taken in order, is equal to the product of the other three.*



Let $\triangle ABC$ be a \triangle , and let AD, BE, CF which intersect at O , meet the opp. sides at D, E, F .

Let the numerical measures of the segments $(BD, DC), (CE, EA), (AF, FB)$ referred to one and the same unit be $(p, p'), (q, q'), (r, r')$ respectively.

To prove that $p \cdot q \cdot r = p' \cdot q' \cdot r'$.

Proof. Draw $BH, CK \perp$ to AD .

Let α, β, γ be the numerical measures of the areas of the \triangle s BOC, COA, AOB respectively.

Now, the \triangle s BHD, CKD are equiangular ;

$$\therefore \frac{BD}{DC} = \frac{BH}{CK}.$$

Also the \triangle s AOB, AOC have the same base AO , and their altitudes are BH, CK respectively ;

$$\therefore \frac{\triangle AOB}{\triangle AOC} = \frac{BH}{CK}.$$

$$\text{Hence, } \frac{BD}{DC} = \frac{\triangle AOB}{\triangle AOC},$$

$$\text{i.e., } \frac{p}{p'} = \frac{\gamma}{\beta} \quad \dots \dots \dots (1)$$

$$\text{Similarly, } \frac{q}{q'} = \frac{\alpha}{\gamma} \quad \dots \dots \dots (2)$$

$$\text{and } \frac{r}{r'} = \frac{\beta}{\alpha} \quad \dots \dots \dots (3)$$

Hence, from (1), (2) and (3),

$$\frac{p}{p'} \cdot \frac{q}{q'} \cdot \frac{r}{r'} = \frac{\gamma}{\beta} \cdot \frac{\alpha}{\gamma} \cdot \frac{\beta}{\alpha} = 1;$$

$$\therefore p \cdot q \cdot r = p' \cdot q' \cdot r'. \quad \text{Q. E. D.}$$

Note 1. This Proposition is known as **Ceva's Theorem**.

Note 2. The converse of this Proposition is the following :

If three straight lines AD, BE, CF drawn from the vertices of a $\triangle ABC$, meet the opposite sides in D, E, F, so that $BD \cdot CE \cdot AF = DC \cdot EA \cdot FB$; then the three straight lines AD, BE, CF are concurrent. It can be proved as follows :—

Let AD, BE intersect at O; and let CO produced meet AB in F'. Let (p, p') , (q, q') , (r, r') , (s, s') be the numerical measures of (BD, DC), (CE, EA), (AF, FB) and (AF', F'B) respectively, referred to one and the same unit.

Then, as before,

$$p \cdot q \cdot s = p' \cdot q' \cdot s'.$$

$$\text{But } p \cdot q \cdot r = p' \cdot q' \cdot r'. \quad (\text{Hyp.})$$

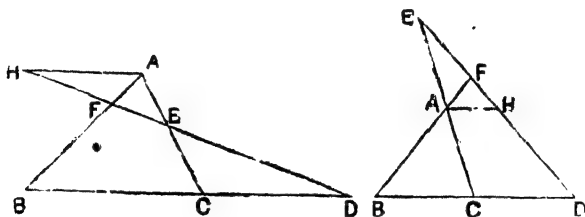
$$\therefore \frac{s}{r} = \frac{s'}{r'}, \text{ and } \therefore \frac{s}{s'} = \frac{r}{r'}.$$

$$\text{Hence, } \frac{AF'}{F'B} = \frac{AF}{FB};$$

i.e., F' and F both divide AB internally (or externally) in the same ratio; \therefore F' coincides with F (Prob. 3, Note 3). Thus, CO produced

passes through **F**, which proves that **AD**, **BE**, **CF** all pass through the same pt. **O**.

4. *If a straight line cuts the sides, or the sides produced, of a triangle, the product of three alternate segments taken in order is equal to the product of the other three.*



Let **ABC** be a \triangle ; and let a str. line cut the sides **BC**, **CA**, **AB**, or these sides produced, at **D**, **E**, **F** respectively.

Let the numerical measures of (**BD**, **DC**), (**CE**, **EA**), (**AF**, **FB**) referred to one and the same unit be (p , p'), (q , q'), (r , r') respectively.

To prove that $p \cdot q \cdot r = p' \cdot q' \cdot r'$.

Proof. Draw **AH** \parallel to **BC**, meeting the str. line **DEF** in **H**; and let m be the numerical measure of **AH** referred to the unit already used.

The two \triangle s **BDF**, **AHF** are equiangular;

$$\therefore \frac{BD}{AH} = \frac{BF}{AF};$$

$$\therefore \frac{p}{m} = \frac{r'}{r} \quad \dots \quad \dots \quad \dots \quad (1)$$

Also the two \triangle s **AHE**, **CDE** are equiangular;

$$\therefore \frac{AH}{CD} = \frac{AE}{EC};$$

$$\therefore \frac{m}{p'} = \frac{q'}{q} \quad \dots \quad \dots \quad \dots \quad \dots \quad (2)$$

Hence, from (1) and (2),

$$\frac{p}{q'} = \frac{r'}{r} \cdot \frac{q'}{q};$$

$$\therefore p \cdot q \cdot r = p' \cdot q' \cdot r'.$$

Q. E. D.

Note 1. This proposition is known as **Menelaus Theorem**.

Note 2. The converse of this proposition is the following :—

If in a triangle, three points are taken, one on each of two sides and one on the produced part of the third side, (or one on the produced part of each of the three sides), so that the product of three alternate segments taken in order is equal to the product of the other three, then the three points are collinear. The first case can be proved as follows :—

In the $\triangle ABC$, let **D** be taken on **BC** produced and **E, F** on **CA** and **AB** respectively, so that $BD \cdot CE \cdot AF = DC \cdot EA \cdot FB$; to prove that the points **D, E, F** are collinear.

Join **DE** and produce it to meet **AB** in **F'**. Let the numerical measures of **(BD, DC), (CE, EA), (AF, FB), (AF', FB')**, referred to one and the same unit, be respectively $(p, p'), (q, q'), (r, r'), (s, s')$.

Then, as before,

$$p \cdot q \cdot s = p' \cdot q' \cdot s'.$$

$$\text{But } p \cdot q \cdot r = p' \cdot q' \cdot r'.$$

(Hyp.)

$$\therefore \frac{s}{r} = \frac{s'}{r'}, \quad \text{and} \quad \therefore \frac{s}{s'} = \frac{r}{r'}.$$

$$\text{Hence, } \frac{AF'}{F'B} = \frac{AF}{FB}$$

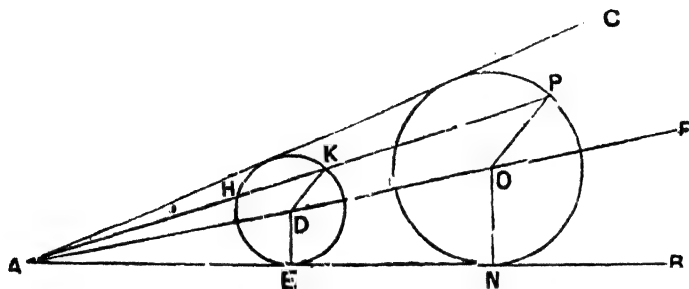
i.e. **F'** and **F** both divide **AB** internally in the same ratio ; \therefore **F'** coincides with **F**.

Hence, **DE** produced passes through **F**.

Q. E. D.

Similarly the other case can be proved.

5. *Construct a circle passing through a given point **P** and touching two given straight lines **AB**, **AC**.*



CONS. Let **AF** be the bisector of the \angle **BAC**; then the centre of the reqd. \odot , being equidistant from **AB** and **AC**, must be on **AF**.

Take any pt. **D** on **AF**, and draw **DE** \perp to **AB**.

With centre **D** and radius **DE** describe a \odot ; it will touch **AB** and **AC**.

Join **AP** cutting the above \odot in **H** and **K**.

Join **KD**; and draw **PO** \parallel to **KD**, meeting **AF** in **O**.

Then **O** is the centre of the reqd. \odot .

Proof. Draw **ON** \perp to **AB**.

Then the \triangle s **AED**, **ANO** are equiangular ;

$$\therefore \frac{ON}{DE} = \frac{OA}{DA}.$$

Also the \triangle s ADK , AOP are equiangular ;

$$\therefore \frac{OA}{DA} = \frac{OP}{DK}.$$

Hence, $\frac{ON}{DE} = \frac{OP}{DK} ;$

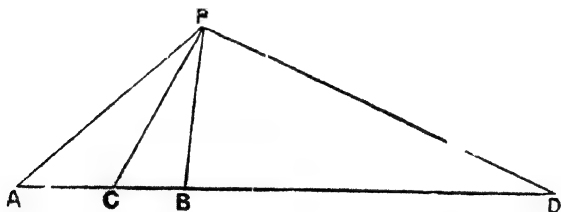
$$\therefore \frac{ON}{OP} = \frac{DE}{DK}.$$

But $DE = DK$, $\therefore ON = OP$.

Hence, the \odot described with O as centre and ON as radius is the reqd. \odot . Q. E. F.

Note. If HD were joined and if PO' were drawn parallel to HD , meeting AF in O' , then O' would be the centre of another circle satisfying the given conditions.

6. *Given the base of a triangle and the ratio of the two sides ; shew that the different positions of the vertex all lie on the circumference of a known circle.*



Let AB be the given base, and let the ratio of the sides be $= \frac{m}{n}$, where m and n are integers.

Let P be any position of the vertex, so that $\frac{AP}{PB} = \frac{m}{n}$.

To prove that P lies on the circumference of a known circle.

* **Proof.** Join PA, PB ; and let PG, PD be the internal and external bisectors of the $\angle APB$, meeting AB in C and D .

Then the $\angle CPD$ is a rt. \angle .

Now, since PC is the internal bisector of the $\angle APB$,

$$\therefore \frac{AC}{CB} = \frac{AP}{PB} = \frac{m}{n};$$

also, since PD is the external bisector of the $\angle APB$,

$$\therefore \frac{AD}{DB} = \frac{AP}{PB} = \frac{m}{n}.$$

Thus, C and D divide AB internally and externally in the ratio of m to n , and are \therefore known pts.

Now, since the $\angle CPD$ is a rt. \angle , the \odot described on CD as diameter will pass through P .

Hence the \odot described on CD as diameter is the \odot on which P as well as every other position of the vertex lie.

Q. E. D.

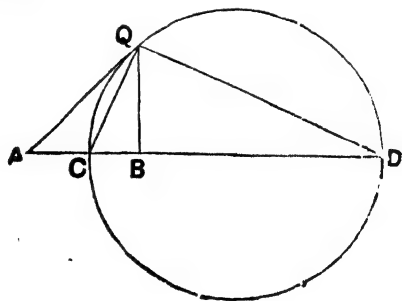
Note. Take *any* point Q on the circle described on CD as diameter; join QA, QB, QC, QD .

Now, $\frac{AC}{CB} = \frac{AD}{DB}$;

each being $= \frac{m}{n}$; and

CD subtends a right angle at Q . Hence, it can be proved, as in Th. 10, Note 1, that QC bisects the angle AQB ; and \therefore

$$\frac{AQ}{QB} = \frac{AC}{CB} = \frac{m}{n}.$$



Hence, **Q** is one of the positions of the vertex of the given triangle.

Now it is clear that *every* position of the vertex of the given triangle is on the circle described on **CD** as diameter, and conversely, *every* point on this circle is a position of the vertex. Hence, we may say that the *locus* of the vertex of the given triangle is the circle described on **CD** as diameter.

EXERCISE (35).

1. **ABCD** is a quadrilateral, and **S** is a point outside it. **A'** is a point in **AS** such that $AA' : A'S = 2 : 1$, **B'**, **C'**, **D'** are points in **SB**, **SC**, **SD** such that **A'B'**, **B'C'**, **C'D'** are parallel to **AB**, **BC**, **CD** respectively. Prove that the area of the figure **A'B'C'D'** is one-ninth of the area of the given quadrilateral.

2. Construct a quadrilateral similar to a given quadrilateral so that each side of the constructed figure may be twice the corresponding side of the given figure.

3. **ABC** is a triangle, **D** and **F** being any two points in **BC**, **AB** respectively. **DEF** is a triangle, the point **E** being on the side of **DF** remote from **B**. If **BE**, or **BE** produced, intersects **CA** in **E'**, and if **E'D'**, **E'F'** be drawn parallel to **ED**, **EF**, meeting **BC**, **AB** in **D'**, **F'** respectively, prove that the triangle **D'E'F'** is similar to the triangle **DEF**.

4. In a given triangle inscribe a triangle similar to another given triangle.

5. Prove that the internal and external *centres of similitude* of two given circles respectively divide the line joining the centres internally and externally in the ratio of the radii of the circles.

6. If a circle touches two given circles, prove that the line joining the points of contact passes through a centre of similitude of the two circles.

7. Prove that the orthocentre and the centroid of a triangle are respectively the external and internal centres of similitude of the circumcircle and nine-point circle of the triangle.

8. Prove that the straight lines which join the vertices of a triangle to the points of contact of the in-circle are concurrent.

9. The in-circle of a triangle ABC touches the sides BC , CA , AB at D , E , F . EF , FD , DE produced meet BC , CA , AB respectively in L , M , N . Prove that L , M , N are collinear.

10. From a given point O a straight line is drawn meet a given circle in P ; in OP a point Q is taken so that OP is to OQ in a given ratio. Find the locus of Q .

APPENDIX.

MODEL PAPERS.

MATRICULATION EXAMINATION.

(1) Compulsory Course.

[Time allowed for answering each paper—1½ hours.]

I.

1. If two straight lines intersect, prove that the vertically opposite angles are equal.

Shew that the bisectors of two vertically opposite angles are in one and the same straight line.

2. At a given point in a given straight line construct an angle equal to a given angle, using Ruler and Compasses.

3. Prove that parallelograms on the same base and of the same altitude are equal in area.

Deduce that the area of a parallelogram is equal to that of a rectangle contained by the base and altitude of the parallelogram.

4. Prove that there is one circle, and one only, which passes through three given points not in a straight line.

Deduce that one circle cannot cut another in more than two points,

5. If two chords of a circle intersect inside the circle prove that the rectangle contained by the parts of the one,

is equal to the rectangle contained by the parts of the other.

If two straight lines AB , CD intersect at P so that $AP \cdot PB = CP \cdot PD$, prove that the four points A , B , C , D are concyclic.

II.

1. Prove that two triangles are equal in every respect if the three sides of one triangle are respectively equal to the three sides of the other.

Prove that the straight line drawn from the vertex of an isosceles triangle to the mid-point of the base is perpendicular to the base.

2. Shew that the locus of a point which is equidistant from two intersecting straight lines consists of the pair of straight lines which bisect the angles between the two given lines.

Deduce that the point of intersection of the bisectors of any two angles of a triangle is equidistant from the three sides of the triangle.

3. Construct a rectangle equal to a given triangle using the necessary instruments. Also prove the construction.

4. Prove that triangles on the same base and of the same altitude are equal in area.

Hence shew that triangles on equal bases and of equal altitudes are equal in area.

5. If the line joining two points subtends equal angles at two other points on the same side of it, prove that the four points lie on a circle.

Shew that triangles standing on the same base, and on the same side of it, with equal vertical angles have their vertices on the circumference of a circle of which the given base is a chord.

III.

1. Prove that the sum of the angles of a triangle is equal to two right angles.

If any side of a triangle is produced, shew that the exterior angle is equal to the sum of the two interior opposite angles.

2. If there are three or more parallel straight lines, and the intercepts made by them on any straight line that cuts them are equal, prove that the corresponding intercepts on any other straight line that cuts them are equal.

If D be the mid-point of the side AB of a triangle ABC, shew that the straight line drawn through D parallel to BC will bisect AC.

3. Shew that the locus of a point which is equidistant from two fixed points is the perpendicular bisector of the straight line joining the two fixed points.

Deduce that the point of intersection of the perpendicular bisectors of any two sides of a triangle is equidistant from the vertices of the triangle.

4. Divide a given straight line into seven equal parts, using the necessary instruments. Give reasons for the construction.

5. Prove that the opposite angles of any quadrilateral inscribed in a circle are supplementary.

If a circle passes through the angular points of a parallelogram, shew that the parallelogram must be a rectangle.

IV.

1. Prove that two right-angled triangles are equal in every respect if they have their hypotenuses equal, and one side of the one equal to one side of the other.

OB and OC are perpendiculars upon two straight lines AB and AC which meet at A : if $OB = OC$, prove that AO is the bisector of the angle BAC.

2. Illustrate and explain the Geometrical theorem corresponding to the identity,

$$(a + b)^2 = a^2 + 2ab + b^2.$$

If P is the mid-point of a straight line AB, shew that $AB^2 = 4AP^2$.

3. Construct a triangle equal in area to any given quadrilateral, using the necessary instruments.

4. Prove that a straight line drawn from the centre of a circle to bisect a chord which is not a diameter, is at right angles to the chord.

Hence shew that a straight line cannot cut a circle in more than two points.

5. If a straight line touch a circle, and from the point of contact a chord be drawn, prove that the angles which this chord makes with the tangent are equal to the angles in the alternate segments.

If ABC is a triangle, and if through B, a straight line BD be drawn on the side of BC remote from A such that

the angle CBD is equal to the angle BAC , shew that BD must be the tangent at B to the circum-circle of the triangle ABC .

V.

1. Prove that any two sides of a triangle are together greater than the third.

Hence shew that the difference of any two sides of a triangle is less than the third side.

2. Illustrate and explain the Geometrical theorem corresponding to the identity $(a - b)^2 = a^2 - 2ab + b^2$.

If C is the mid-point of a given straight line AB , and if D is any point in CB , shew that $AD^2 + DB^2 = 2AC^2 + 2CD^2$.

3. Construct a rectangle equal to any given quadrilateral, using the necessary instruments.

4. Prove that equal chords of circle are equidistant from the centre. Hence shew that in a circle the middle points of equal chords are concyclic.

5. Prove that the angle in a semi-circle is a right angle.

Hence shew how to construct a tangent to a circle from any point outside it.

VI.

1. If the sides of a convex polygon are produced in order, prove that the sum of the angles so formed is equal to four right angles.

Hence find the magnitude of an angle of a regular pentagon.

2. Define *parallel straight lines*. If a straight line cutting two other straight lines makes the alternate angles equal, prove that those two straight lines are parallel.

Hence shew that if two straight lines are perpendicular to the same straight line, they are parallel to one another.

3. Construct a tangent to a given circle from a given point outside it, using the necessary instruments.

4. Prove that the tangent at any point of a circle is perpendicular to the radius through the point.

If from the point of contact of a tangent to a circle a straight line be drawn perpendicular to the tangent, prove that it must pass through the centre of the circle.

5. If the opposite angles of a quadrilateral are supplementary, prove that a circle can be described through its angular points.

The side AB of a quadrilateral ABCD is produced to E; if the angle CBE be equal to the angle ADC, shew that the four points A, B, C, D are concyclic.

VII.

1. If a straight line cuts two parallel straight lines, prove that the alternate angles are equal.

Hence shew that if a straight line is perpendicular to one of two parallel straight lines, it is also perpendicular to the other.

2. If the square described on one side of a triangle be equal to the sum of the squares described on the other two sides, prove that the angle contained by those two sides is a right angle.

Hence show that a triangle whose sides are respectively 28, 45 and 53 units of length, is a right-angled triangle.

3. Take a triangle; bisect any two of its angles; from the point where the two bisectors meet, draw perpendiculars to the sides of the triangle. Verify by actual measurement that these three perpendiculars are equal in length.

4. If two tangents are drawn to a circle from an external point, prove that they are equal and that they subtend equal angles at the centre of the circle.

Two tangents are drawn to a circle at the extremities of a diameter; shew that the intercept between them of any third tangent subtends a constant angle at the centre.

5. If two chords of a circle intersect outside the circle, prove that the rectangle contained by the parts of the one is equal to the rectangle contained by the parts of the other.

If from any point P without a circle two straight lines be drawn, one of which cuts the circle in A and B , and the other touches it at T , prove that $PA \cdot PB = PT^2$.

VIII.

1. If two sides of a triangle are unequal, prove that the greater side has the greater angle opposite to it.

Hence if one angle of a triangle be greater than another prove that the side opposite to the greater angle is greater than the side opposite to the less.

2. Prove that in a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the sides containing the right angle.

Shew how to construct a square whose area is treble that of a given square.

3. Take a triangle; draw perpendiculars to any two of its sides at their middle points; join the angular points to the point of intersection of these two perpendiculars. Verify, by actual measurement, that the three straight lines so drawn are equal in length.

4. Prove that in a circle the perpendicular drawn from the centre to a chord bisects the chord.

Hence shew that chords which are equidistant from the centre are equal.

5. If two circles touch, prove that the point of contact lies on the straight line through the centres.

Two circles whose centres are O and O' touch each other externally at P ; APB is a straight line through P meeting the circles again at A and B . Prove that OA is parallel to $O'B$.

IX.

1. Prove that two triangles are equal in every respect if two angles and a side of the one triangle are respectively equal two angles and the corresponding side of the other.

If the bisector of the vertical angle of a triangle is also perpendicular to the base, shew that the triangle is isosceles.

2. Prove that, in an obtuse-angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the other two sides together with twice the rectangle contained by one of these two sides and the projection on it of the other

If the numerical measures of the sides of a triangle are 7, 9 and 11, shew that the angle opposite to the greatest side is neither right nor obtuse.

3. Through a given point construct a straight line parallel to a given straight line, using Ruler and Compasses. Give reasons for the construction.

4. Prove that, in equal circles, if two arcs subtend equal angles at the centre, they are equal.

Hence shew that in equal circles, sectors having equal angles are equal.

5. Prove that the angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.

Shew that, in equal circles, arcs which subtend equal angles at the circumferences are equal.

X.

1. If two sides of a triangle are equal, prove that the angles opposite to these sides are equal.

Shew that three angles of an equilateral triangle are equal to one another.

2. Construct a parallelogram, with Ruler and Compasses; and verify, by actual measurement, that its diagonals bisect each other.

3. Prove that in any triangle the square on the side opposite to an acute angle is equal to the sum of the squares on the other two sides diminished by twice the rectangle contained by one of these two sides and the projection on it of the other.

The numerical measures of the sides of a triangle are 8, 13 and 17; shew that the angle opposite to the greatest side is neither right nor acute.

4. In two equal circles, if there be two equal arcs one in each, prove that the chords of these two arcs are equal.

Hence shew that in the same circle chords of any two equal arcs are equal.

5. Prove that angles in the same segment of a circle are equal.

If two triangles standing on the same base, on the same side of it, have equal vertical angles, shew that both the triangles will have the same circum-circle.

(2) Optional Course

[Time allowed for answering each Paper—1½ hours.]

I.

1. Prove that the sum of the squares on any two sides of a triangle is equal to twice the square on half the third side together with twice the square on the median that bisects third side.

ABCD is a rectangle ; and P is any point, either inside or outside it. Prove that $PA^2 + PC^2 = PB^2 + PD^2$.

2. Construct an equilateral triangle in a given circle.

3. AD, BE are perpendiculars from the vertices A and B of a triangle on the opposite sides ; BF is perpendicular to ED or ED produced. Prove that the angle FBD is equal to the angle EBA.

4. If two triangles are equiangular, prove that their corresponding sides are proportional.

In a right-angled triangle, if a perpendicular be drawn from the right-angle to the hypotenuse, prove that the triangles on each side of it are similar to the whole triangle and to one another.

II.

1. Prove that the tangent at any point of a circle is perpendicular to the radius through the point.

If two circles are concentric, prove that all chords of the outer circle which touch the inner are equal.

2. Construct a circle, with a given radius, touching two given straight lines.

3. If two triangles standing on equal bases have their vertical angles supplementary, prove that their circum-circles are equal.

4. When are two polygons said to be *similar*? Illustrate your meaning by a diagram.

If a polygon is divided into triangles by lines joining a given point to its vertices, prove that any similar polygon can be divided into corresponding similar triangles.

III.

1. If two tangents are drawn to a circle from an external point, prove that they are equal and that they subtend equal angles at the centre of the circle.

If a quadrilateral be circumscribed about a circle, prove that the angles subtended at the centre of the circle by any two opposite sides of the figure are together equal to two right angles.

2. Construct a circle passing through a given point and touching a given circle at a given point.

3. Prove that the distance of each vertex of a triangle from the ortho-centre is double the distance of the circum-centre from the opposite side.

4. Prove that the internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.

ABCD is a quadrilateral, and the bisectors of the angles A and C meet on the diagonal BD. Prove that the bisectors of the angles B and D meet on AC.

IV.

1. If a straight line touch a circle, and from the point of contact a chord be drawn, prove that the angles which this chord makes with the tangent are equal to the angles in the alternate segments.

Two circles intersect at A and B, and through P, any point in the circumference of one of them, the chords PA and PB are drawn to cut the other circle at C and D. Prove that CD is parallel to the tangent at P.

2. Construct a circle which shall touch a given circle, have its centre in a given straight line and pass through a given point in the given straight line.

3. AB is any chord of a given circle, and AD is a tangent to the circle at A. DPQ is a straight line parallel to AB, meeting the circumference in P and Q. Prove that the triangle PAD is equiangular to the triangle QAB.

4. If a straight line is drawn parallel to one of a triangle, prove that it cuts the other two sides proportionally.

ABCD is a trapezium of which AB, DC are the parallel sides. If any straight line EF parallel to AB cuts AD, BC in E and F respectively, prove that

$$AE : ED = BF : FC.$$

V.

1. Prove that the opposite angles of any quadrilateral inscribed in a circle are supplementary.

Through each of the points of intersection of two circles, right lines are drawn, cutting one circle in A and B, and the other in C and D. Prove that AB is parallel to CD.

2. Construct three circles touching each other and having their centres at three given points which are not collinear.

3. AB is an arc of a circle and P its middle point ; PM is any chord meeting the arc of the 'opposite segment in M and AB in N . Prove that $PM.PN = PA^2$.

4. If two triangles have their sides proportional, prove that they are also equiangular, having those angles equal which are opposite to corresponding sides.

State the conditions of similarity of two rectilineal figures, and shew that in the case of two triangles one of those conditions is sufficient.

VI.

1. Prove that the angle in a semi-circle is a right angle.

AD is a diameter of a circle circumscribed about any given triangle ABC , and L is the point of intersection of perpendiculars drawn from B and C on the opposite sides of the triangle. Prove that LD and BC bisect each other.

2. Construct a circle having a given radius and touching a given circle and a given straight line.

3. AB is a diameter of a circle, and AF any chord. C is any point in AB , and through C a straight line is drawn at right angles to AB , meeting AF , produced if necessary, at G and the circumference at D , prove that

$$FA.AG = BA.AC = AD^2.$$

4. Prove that the ratio of the areas of two similar triangles is equal to the ratio of the squares on the corresponding sides.

Shew that any given triangle is divided into four equal parts by the straight lines joining the mid-points of its sides.

VII.

1. Prove that the sum of the squares on the four sides of a quadrilateral is equal to the sum of squares on its diagonals together with four times the square on the line joining the mid-points of the diagonals.

2. If two chords of a circle intersect outside the circle, prove that the rectangle contained by the parts of the one is equal to the rectangle contained by the parts of the other.

AO, BO are radii of a circle at right angles to each other. ACD is a straight line meeting OB in C and the circle in D. Prove that

$$AC \cdot AD = 2 OB^2.$$

3. Construct a regular hexagon about a given circle.

4. Prove that the ratio of the areas of two similar quadrilaterals is equal to the ratio of the squares on the corresponding sides.

VIII.

1. ABC is a triangle, and the exterior angles at B, C are bisected by lines BD, CD respectively, meeting in D. Prove that the angle BDC and half the angle BAC make up a right angle.

2. If two circles touch, prove that the point of contact lies on the straight line through the centres.

If the centres of two circles which touch each other externally be fixed, shew that the common tangents of the circles will touch another circle of which the straight line joining the fixed centres is the diameter.

3. Construct an equilateral triangle about a given circle.

4. If two triangles have one angle of the one equal to one angle of the other, and the sides about these equal angles proportional, prove that the triangles are similar.

If two straight lines AB, CD intersect at P, so that $AP : PC = PD : PB$, shew that the triangles APD, CPB are similar.

IX.

1. State and prove the Geometrical theorem corresponding to the identity $(a - b)^2 = a^2 - 2ab + b^2$.

If C is the mid-point of a given straight line AB, and if D is any point in CB produced, prove that $AD^2 + DB^2 = 2AC^2 + 2CD^2$.

2. AB is a diameter of a circle, and C, D are any two points on the semi-circle ACDB. The chords AD, BC intersect at E. Prove that the circum-circle of the triangle CDE will cut the given circle at right angles.

3. Construct a circle touching a given straight line at a given point, such that the tangents drawn to it from two given points in the straight line may be parallel.

4. If two triangles be equiangular, prove that their circum-radii are in the ratio of any pair of corresponding sides.

Hence shew that the nine-point radius of any triangle is half the circum-radius.

X.

1. Shew that the angular points of regular pentagon are concyclic.

2. Prove that angles in the same segment of a circle are equal.

AB is a common chord of two circles; through C, any point on one of the circles, straight line CAD, CBE are drawn, terminated by the circumference of the other circle. Shew that the arc DE is invariable.

3. Bisect a triangle by a line drawn parallel to one of its sides.

4. A and B are two fixed points, and P is a variable point such that $PA : PB = 3 : 2$. Prove that P lies on the circumference of a fixed circle.

ANSWERS AND HINTS.

Exercise (1). [Pages 19—21.]

1. *A solid,* 2. *A solid,* 3. *Length.*
4. *A surface*; because it has length and breadth but *no thickness.*
5. *Only when* the two lines *coincide.*
6. The points **A** and **B** are in *each* plane; hence the straight line **AB** also lies wholly in *each* plane, and is therefore common to the two planes.
7. Because then they entirely coincide.
8. Because when the paper is folded the two parts **CA** and **CB** coincide and are therefore *equal.*
9. When their vertices coincide and the arms of one fall upon those of the other.
10. When the paper is folded the angle **COA** coincides with the angle **COB** and is therefore equal to it; hence **OC** is perpendicular to **AB**.
11. Let **AO** be produced to any point **B**; when the revolving line falls upon **OB** it has turned through two right angles, and it turns through two right angles more when it rotates from the position **OB** into the position **OA**. Hence it altogether turns through *four* right angles.
12. With **A** as centre and with a radius equal to **CD** draw a circle, cutting **AB** at **E**. Then **AE** is the required part; for **AE** is a radius of the circle, and is therefore equal to **CD**.

13. Take any other point D and draw a straight line from A to D. With A as centre and with a radius equal to BC draw a circle cutting AD or AD produced (if necessary) at E. Then AE is the required straight line; for AE is the radius of the circle and is therefore equal to BC, and it is also drawn from A.

14. Because when the paper is folded the angle COA coincides with the angle COB and is therefore equal to it.

Exercise (4). [Pages 43—44.]

7. Produce BD to meet AC at E. 8. Produce BD to meet AC at E. Then, it can be shewn that $AB + AC > BE + EC$.

9. In proving the first part, apply the result of the last example.

Exercise (6). [Pages 57—58.]

2. Drawing a figure, it may be shewn that the greatest angle is equal to its supplement. 3. The greatest angle may be shewn to be greater than its supplement, and \therefore it exceeds a rt. angle. 4. A diagonal divides it into two triangles. 7. 144° 9. 36° 13. The $\angle BCD$ may be shewn to be equal to its supplement.

Exercise (8.) [Pages 73—74.]

5. Join AA' , BB' , CC' . 9. Produce DE to F making $EF = DE$ and join FC; then FC may be shewn to be equal and parallel to DB.

Exercise (11). [Pages 110—112.]

1. Produce AB to E , making $AE = AC$; join ED . Then $CD = ED$. 2. Produce AD to E , making $DE = AD$. Then $AC = BE$. 4. ED is \parallel to BF . Now apply Th. 24, Cor. 1. 12. Draw $DP \parallel$ to CA , meeting AB in P ; take $PG = PA$. 15. Draw $BH \parallel$ to DE , meeting AP produced in H , draw $CK \parallel$ to DE , meeting AP in K .

Exercise (12). [Page 117.]

7. 12.

Exercise (13). [Pages 124—125.]

3. Through E draw a str. line \parallel to AB . 10. 4.9 sq. inches. 11. 2.5 inches. 12. 5.5 inches.

Exercise (14). [Pages 129—130.]

5. 13 miles. 6. 35 feet. 7. 1 foot.

8. At one extremity of the given line make an \angle equal to half a right \angle . 9. At one end of the given line draw a \perp to it equal to a side of the given square. 12. Draw $BD \perp$ to BC and $= BA$; join DC . Now apply Th. 21, Bk. I.

Exercise (15). [Pages 138—140.]

1. The square on any given line = four times the square on half the line. 2. The sq. on any given line = nine times the sq. on one-third of the line. 6. If a given str. line AB be divided into any three parts at C and D , then $AD \cdot CB = AC \cdot DB + CD \cdot AB$. 7. If E is the mid-pt. of BC , $AE = EB$ or EC . 8. If E is the mid-pt. of BC , then from the given condition we have

$EA^2 - ED^2 = EC^2 - ED^2$. 10. Apply Theorems 10 and 11.
11. Apply Theorems 10 and 11.

Exercise (16). [Pages 143—145.]

8. The circumference of a circle whose centre is the mid-pt. of AB and radius = 4 inches. 11. If ABCD is the par^m, draw CE \parallel to BD, meeting AD produced in E.

Exercise (17). [Pages 151—152.]

10. Draw BH \perp to AB and = a side of the given square. Join AH. Make $\angle AHF = \angle HAF$. 11. Construct a square = the given rectangle, and then proceed as in the last problem.

Exercise (18). [Pages 160—162.]

2. O is the centroid of the $\triangle ABC$. 6. If AD be \perp to BC, it may be shewn that $AC = 2CD$. 9. If AB is the given str. line, make $\angle BAC = 45^\circ$ and $\angle ABC = \frac{1}{2}$ of 45° . Draw CD \perp to AC, meeting AB in D. 14. 84 units of area; 8. 15. 324 units of area. 18. If C is the mid-pt. of AB and PD is \perp to AB, it is easy to see that $PA^2 - PB^2 = 2AB \cdot CD$. 21. The centroid of the \triangle .

Exercise (20). [Page 183.]

4. Draw PM \perp to AB, and produce PM to Q making MQ = PM. Then AP = AQ and BP = BQ.

Exercise (21). [Page 193.]

2. Construct an equilat. $\triangle ABC$ each of whose sides 2 inches; produce BC to D, making $CD = BC$. Then will be equal to a side of the reqd. \triangle .

6. If O be the centre, the $\angle AOC$ is greater than the $\angle AOD$.

Exercise (23). [Pages 209—210.]

9. It may be shewn that the $\angle APB = 90^\circ + \frac{1}{2}$ the $\angle ACB$.

Exercise (25). [Pages 225—226.]

11. The $\angle ACD =$ the $\angle CBD$, each being complementary to the $\angle BCD$.

Exercise (26). [Page 262.]

6. 120° .

Exercise (27). [Pages 293—299.]

17. On the given side construct a segment containing an \angle , equal to $\frac{1}{2}$ the given \angle .

18. C is the mid-pt. of the arc PQ .

24. If P be on the arc AC , from PB cut off $PD = PC$.

25. EF may be shewn to be \parallel to the tangent at A .

26. See Prop. 14.

27. Apply Prop. 14.

29. If O be the centre of the given \odot , it may be shewn that OC touches the $\odot CDE$.

32. See Prop. 1.

33. See Prop. 10.

35. See Prop. 8. It is easy to see that OE touches the $\odot SEQ$.

38. $B'C'$ is \parallel to BC and also to QR .

39. Apply Prop. 10.

41. The distance of either common tangent from the mid-point of the line joining the centres is = half the sum of the radii of the \odot s.

42. The fixed pt. is the pt. where the bisector of the $\angle BAC$ meets the circum- \odot of the $\triangle ABC$. If O be this pt., it may be shewn that the $\angle POQ = \angle BOC$.

45. If PD be \perp to BC , D is the second pt. where the circles on PB, PC as diameters intersect.

46. If A be the fixed pt. and $AK \perp$ to the fixed || str. line, then the \odot with A as centre and AK as radius will be the fixed \odot reqd.

47. See Prop. 12, Note 2.

48. See Prop. 1.

49. In the figure of Prop. 15 it is easy to see that the nine-pt. \odot touches the \odot whose centre is A' and radius = $A'G$.

50. Taking 1 inches = h miles, radius of the earth = R miles and the reqd. distance = X miles, it is easy to shew that $X^2 = (2R + h)h = 2Rh$, neglecting h^2 which is *very small* compared with $2Rh$.

Exercise (28). [Pages 317—318.]

1. 15. 7. The ratio of A to B is equal to the ratio of C to D .

Exercise (32). [Page 317.]

6. Apply Addendo.

Exercise (33). [Pages 321—322.]

3. The two \triangle s ABD, APC are equiangular.

Exercise (34). [Pages 329—330.]

5. See Prob. 4. 6. See Prob. 4. 9. Divide AB at E so that $AE : AB = 1 : 3$, and then find PQ so that $AE : PQ = PQ : AB$.

10. The ratio of the segments into which the base is divided by the bisector of the vertical \angle is known.

Exercise (35). [Pages 370—371.]

2. See the last example. 4. See the last example. 7. See Bk. III, Sec. IV. Prop. 15. 9. Apply Prop. 4 to each of the three lines EFL, DFM, DEN.
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